SCHWARZ REFLECTION GEOMETRY II: LOCAL AND GLOBAL BEHAVIOR OF THE EXPONENTIAL MAP

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Abstract. A local normal form is obtained for geodesics in the space $\Lambda = \{\Gamma\}$ of analytic Jordan curves in the extended complex plane with symmetric space multiplication $\Gamma_1 \cdot \Gamma_2$ defined by Schwarzian reflection of Γ_2 in Γ_1. Local geometric features of $(\Lambda, \cdot)$ will be seen to reflect primarily the structure of the Witt algebra, while issues of global behavior of the exponential map will be viewed in the context of conformal mapping theory.

1. Introduction

Geometry springs from many sources: *Symmetry* defines the homogeneous geometries of Möbius, Lie and Klein; *measurement* is the essence of the inhomogeneous spaces of Riemann and Einstein; *connections* are basic to the more general geometries of Cartan and to twentieth century developments such as gauge theory.

A more specialized source of geometric structure, *reflection* generates a subclass of the former, homogeneous geometries; namely, a *symmetric space* may be defined simply as a manifold-with-multiplication $p \cdot q =$*reflection of* q *in* p (such that, for any point p, left multiplication $s_p = p\cdot$ is an involutive automorphism with isolated fixed point p). As developed in [Loos], all elements of symmetric space geometry—the homogeneous and (when present) metric structures, the connection, curvature, geodesics, etc.—then follow as consequences of such multiplication.

The present work is the second part of an ongoing exploration of *Schwarzian reflection* as a source of geometry on an infinite dimensional space Λ comprised of analytic curves in the complex plane. The development is not straightforward because many standard symmetric space constructions are generally valid only in the finite dimensional setting. For instance, heuristics tempt one to view the space of unparametrized, analytic, closed curves as a quotient space $\Lambda = G/H$—analytic embeddings of the unit circle S^1 into $\mathbb{C}$ modulo analytic diffeomorphisms of S^1 (reparametrizations)—but, unfortunately, the analytic embeddings do not form a group (put another way, the only candidate for G would be the complexification $H_{\mathbb{C}}$ of the diffeomorphism group H, which does not exist). Nevertheless, as shown in *Schwarz Reflection Geometry I*, many elements of symmetric space theory apply nicely to Λ on a formal level and yield locally meaningful equations. Thus, one of our underlying goals is to see how far "standard" geometric interpretations of solutions to such equations may be pushed, as well as to explore "nonstandard" interpretations for *untoward behavior* exhibited by Λ.

Our point of departure in [C-L] was to represent elements of Λ via *Schwarz functions*; we recall from [Davis] that the Schwarz function of an analytic curve Γ is the unique holomorphic function $S(z)$ for which Γ is the locus of the equation $\bar{z} = S(z)$. A key point was that Schwarz functions could be viewed as *symmetric elements* (see [Loos]) in the abstract setting of a group with involution (G, σ), and that symmetric space formalism could subsequently be invoked. (Here, G consists of conformal maps, $\sigma(g)(z) = \overline{g(\bar{z})}$, and the operations of composition, inversion, and involution are only locally defined.) Further, the use of Schwarz functions yielded concrete analytic expressions

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for basic geometric objects and operations which were seen in [C-L] to be remarkably tractable. In particular, a compact formula for the canonical connection on Λ was derived and led to a simple geodesic equation—a conformally invariant, quadratic, second order PDE for a time-dependent Schwarz function $S(t,z)$, implicitly describing an evolving analytic curve Γ_t. Moreover, the latter equation was shown to coincide with a continuous limit of iterated Schwarzian reflection of analytic curves. Finally, such motion was reduced, locally, to the flow of a holomorphic vectorfield $W = w(z)\frac{\partial}{\partial z}$, and data associated with the dual meromorphic differential $\omega = \frac{dz}{w(z)}$ were seen to determine conformal-geometric properties of the evolving curve.

In the present paper, we continue to develop major themes of Schwarz reflection geometry initiated in [C-L], systematically addressing some of the main problems posed there. For example, we obtain a comprehensive local description of all geodesics, up to conformal equivalence, near a given initial curve $\Gamma_\circ$, and generate representative examples. More specifically, by considering the action of the group of analytic circle diffeomorphisms $H = \mathrm{Diff}^\omega_+(S^1)$ on (*normal*) analytic vectorfields W along S^1, we obtain the following main result on short–time behavior of geodesics departing from $\Gamma_\circ = S^1$:

Theorem 1. *For $W = w(z)\frac{\partial}{\partial z} \in T_{\Gamma_\circ}\Lambda$, the geodesic departing from $\Gamma_\circ$ with velocity W is representable in the form $Exp(tW) = h(Exp(tV))$, for small $|t|$, with $h \in H$ and rational vectorfield $V = \frac{a_m z^m + \cdots + a_0}{b_n z^n + \cdots + b_0}\frac{\partial}{\partial z}$ which can be constructed explicitly using meromorphic data of $\omega = \frac{dz}{w(z)}$, the dual differential to W.*

The result is very simple for the special case of a vectorfield W that is non-vanishing along S^1: W is equivalent to a uniform field $\frac{i}{c}\frac{\partial}{\partial\theta}$, with real parameter $c = \frac{1}{2\pi}\int_{S^1} i\omega$ characterizing the orbits $H\frac{i}{c}\frac{\partial}{\partial\theta} \simeq H/S^1$. The curves comprising the *maximal geodesic* $\Gamma_t = Exp(tW)$ for $|t| < m$ foliate a *singular ring domain* $\mathcal{D}$ in the extended complex plane. Allowing W to vary over any given orbit, the maximal geodesics $Exp(tW)$ define a geodesic foliation of $\Lambda_\pm \subset \Lambda$, the space of all simple closed analytic curves in the extended complex plane disjoint from S^1.

Such conclusions may be drawn readily from the theory of conformal mapping and moduli of ring domains, which also can be used to derive information about the time m required for a given such geodesic $Exp(tW)$ to run out of Λ. In terms of locations of singularities of the (*normalized*) vectorfield $\hat{W}$, we obtain a sharp upper bound on $m(\hat{W})$ —see Proposition 3. The sharpness of our estimate, given by a ratio of elliptic integrals, follows from the solution to an extremal problem which is well-known in quasiconformal mapping theory.

Turning to the general case, $W \in T_{\Gamma_\circ}\Lambda$, it is natural to draw the connection between $Exp(tW)$ and the established circle of ideas involving meromorphic flows, trajectories of quadratic differentials, foliated flat geometries (see, e.g., [Mucino-Raymundo]); these ideas are incorporated into our examples, which are integral to the development of the theoretical results presented here. However, such standard methods do not appear to give one a handle on such questions as: *which curves $\Gamma \in \Lambda$ intersecting $\Gamma_\circ$ lie in the image of the exponential map at $\Gamma_\circ$?*

On the other hand, the structure of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus i\mathfrak{h}$ of analytic vectorfields along S^1 is directly relevant to questions of local geometry of Λ. For example, the bracket relations for the *Witt algebra* easily imply that Λ does not have *pseudo-Riemannian* structure (Proposition 4). Further, the proof of Theorem 1 is related to the classification of Adjoint orbits of H. To be precise, the following result provides the basis for explicit construction of normal forms for the vectorfields $W \in T_{\Gamma_\circ}\Lambda$ and locally representative examples of geodesics $Exp(tW)$ (using data $\mu_j, \nu_j, \sigma_j, \mathcal{I}_k, \mathcal{I}$ to be described later):

Theorem 2. *i) Two real, meromorphic differentials on S^1 are equivalent if and only if they have identical order-residue-polarity-period data, $\mu_j, \nu_j, \sigma_j, \mathcal{I}_k, \mathcal{I}$, for suitable counterclockwise orderings of singularities.*

ii) For such differentials with no zeros on S^1, all equivalence classes are represented by restriction to S^1 of rational differentials on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, as the latter achieve arbitrary data $\mu_j, \nu_j, \sigma_j, \mathcal{I}$, subject only to the polarity constraints (PC) and, when there are no poles, $\mathcal{I} \neq 0$.

iii) The Adjoint orbits $Ad_H A$ are characterized by data of the dual meromorphic differentials as in ii); in particular, each is represented by restriction to S^1 of a rational vectorfield on $\hat{\mathbb{C}}$.

Our concrete examples of geodesics not only serve to illustrate the above results, but also raise questions of interpretation which need to be addressed for further development of the subject. In particular, in various ways, our examples bring focus to the question: *how to address the lack of an actual Lie group $G = H_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$?*

Thus, with a view towards a more global theory, we explore issues of apparent multivaluedness and push analytic continuation beyond the point at which our (provisional) notions of Schwarz reflection geometry break down. In this context, Example 1 exposes the relation to trajectory structure of quadratic differentials and foliated flat geometries as *not entirely clear-cut*; Example 3 displays geodesics with topological transitions involving singular and multicomponent curves $\Gamma \notin \Lambda$ (but see Remark 2 for explanation of all other graphical examples); Example 4 admits interpretation as *closed geodesic* foliating a genus one Riemann surface (Remark 3); Example 6 describes geodesics $Exp(tW_n)$ determined by Witt algebra basis elements via "covering map symmetries" $h \notin H$ reflected in the structure of $\mathfrak{h}$.

Some final notes on the contents and organization of the paper may be helpful. Elements of Schwarz reflection geometry introduced in [C-L] are briefly reviewed and extended in §§2,3. We begin by *relativizing* the notion of Schwarz function, allowing an arbitrary analytic curve to be used as *base point* $\Gamma_\circ \in \Lambda$. Regarding the resulting functions $S = S_{\Gamma_\circ}$ as *local coordinates* on Λ, we discuss transformation rules, invariance properties, and variations of curves Γ_t and corresponding Schwarz functions $S(t,z)$. We give equations for the canonical connection on $(\Lambda, \cdot)$, geodesics (Equation 7), and their *canonical parametrization* (Equation 11). In §4 we describe the connection between geodesics in Λ and trajectories of quadratic differentials. The main contents of §5–§7 on ring domains, the isotropy representation λ_*, and geodesics generated by rational vectorfields have already been mentioned above. The last of these sections presents only the "constructive part" of the proof of Theorem 1, as it directly supports examples; the underlying theoretical framework and results are deferred to §8–§10 so as not to interrupt the main flow of ideas of Schwarz reflection geometry. Specifically, in §8 we discuss analytic diffeomorphisms of the circle; in §9, we consider *real meromorphic differentials on* S^1, and describe their invariants with respect to the action of H by pull-back; finally, §10 treats the equivalence problem for such differentials, and completes the proofs of Theorems 1, 2.

2. Analytic Jordan curves and their Schwarz functions

The next paragraphs introduce notation and recall some facts about real analytic curves; further details may be found in [Ahlfors], Ch. V, §1.6. An *analytic arc* or *curve* $\Gamma = \gamma([a,b]) \subset \mathbb{C}$ has a parametrization $\gamma(t) = x(t) + iy(t)$ by real analytic functions $x, y : [a,b] \to \mathbb{R}$. One normally assumes Γ to be *regular* in the sense that γ may be chosen to have non-vanishing velocity $\gamma'(t) \neq 0$. For a *Jordan arc*, γ is one-to-one and Γ may be regarded as the image of $I = [a,b]$ under a conformal map defined by analytic continuation of γ to a symmetric neighborhood $U = U^* \subset \mathbb{C}$ of I; here, $U^* = \{z^* = \bar{z} : z \in U\}$. More specifically, one may replace t by $t + i\tau$ in each local power series representation $\gamma = \sum_{j=0}^{\infty} a_j (t - t_0)^j, t_0 \in I$ and restrict to $|t - t_0| < r$ for sufficiently small $r > 0$, to obtain such a conformal extension $\gamma : U \to \mathbb{C}$.

In the case of a *closed* Jordan curve, we regard Γ instead as the image of the unit circle under a conformal mapping $\gamma : U \to \mathbb{C}$ defined on a symmetric neighborhood of S^1. That is, $U = U^* \subset \mathbb{C} \setminus \{0\}$ where $U^* = \{z^* = 1/\bar{z} : z \in U\}$. It will not cause confusion to use the same notation $b(z) = z^*$ for the *conjugate* (or *reflection*) of a point z with respect to $\mathbb{R}$ or S^1, depending on which of the two cases is considered; likewise for other notation just introduced.

Now consider the image $V = \gamma(U)$ and let $\beta = \beta_\Gamma : V \to V$ be defined by $\beta(z) = \gamma((\gamma^{-1}(z))^*) = \gamma \circ b \circ \gamma^{-1}(z)$. Then β is antiholomorphic, and restricts to the identity mapping on Γ. In fact, these two properties locally characterize β; thus, at each $z_0 \in \Gamma$, the germ of *Schwarzian* (or *antiholomorphic*) *reflection* in Γ is well-defined by the above formula. Likewise, $\beta = \beta^{-1}$ inherits involutivity from the basic reflection b.

Reflection of points in Γ induces *conjugation* (or *reflection*) *of functions* with respect to Γ. That is, a holomorphic function with suitable domain and range yields a new holomorphic function by the operation $f \mapsto \sigma(f) = \beta \circ f \circ \beta$. Most familiar is the case $\Gamma = \mathbb{R}$, allowing *any* function to be conjugated; if $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$ is holomorphic on U, then $\sigma(f)(z) = \overline{f(\bar{z})} = \sum_{j=0}^{\infty} \bar{a}_j (z - \bar{z}_0)^j$ is holomorphic on U^*. When f happens to take real values on the real axis, f is *self-conjugate*, $\sigma(f) = f$, and the *Schwarz symmetry principle* holds: $w = f(z)$ maps a symmetric pair $\{z, z^*\}$ to a symmetric pair $\{w, w^*\}$. (The symmetry principle may be regarded as the basis of the well-known *Schwarz reflection principle*, which we will require in §5; the holomorphic extension provided by this classical result, as usually stated, is self-conjugate *by construction*.)

Similar remarks apply to the case $\Gamma = S^1$. For a general analytic curve Γ, conjugation may be applied only to a smaller class of functions, as β itself is only locally defined. However, the allowable class includes at least holomorphic functions f which are defined, and sufficiently close to the identity, along Γ; this observation will suffice to make sense of our constructions, below, which apply to the following situation. An arbitrarily chosen *base curve* $\Gamma_\circ$ will play the role of the initial reflecting curve, while Γ will now denote a second, nearby curve. Such a curve may be parametrized by $\gamma : \Gamma_\circ \to \Gamma$, which is close to the identity on $\Gamma_\circ$ and extends to a conformal mapping which may be allowed to play the role of f, the function to be conjugated. We note that γ will *never* be self-conjugate—that is, assuming Γ is close but not identical to $\Gamma_\circ$. The context just described is thus complementary to that of the Schwarz symmetry (or reflection) principle, where the operation $f \mapsto \sigma(f)$ is, in itself, *uninteresting*!

The preceding paragraphs pave the way for the introduction of *Schwarz functions relative to a base curve* $\Gamma_\circ$. The general definition is also guided by a simple symmetric space recipe: If G is a Lie group with involution σ, the formula $p{\cdot}q = p\sigma(q)p = pq^{-1}p$ defines a symmetric space multiplication (in the sense of [Loos]) on the subset of *symmetric elements* $G_\sigma = \{\sigma(g)g^{-1} : g \in G\} \subset G$. In particular, Schwarzian reflection across an embedded analytic curve $\Gamma_\circ$ formally defines such an involutive automorphism σ with respect to composition in the pseudogroup G consisting of local conformal maps, and one may therefore expect a symmetric space geometry to result as above.

Specifically, to a conformally parametrized curve $\gamma : \Gamma_\circ \to \Gamma$ we associate a holomorphic function $S = S_{\Gamma_\circ \Gamma}$ defined near Γ, *the Schwarz function of* Γ *relative to* $\Gamma_\circ$:

$$S = \sigma(\gamma) \circ \gamma^{-1} \tag{1}$$

(Fixing the real axis as base curve $\Gamma_\circ = \mathbb{R}$ yields the standard notion of Schwarz function as defined in [Davis]; this was the definition used in [C-L].) We list a few properties of S which follow as formal consequences of Equation 1 (valid under suitable assumptions on domains, etc.): a) S does not depend on the parametrization γ of Γ, b) $S_{\Gamma_\circ \Gamma_\circ} = Id$, c) $S^{-1} = \sigma(S)$, d) $\beta \circ S$ gives antiholomorphic reflection across Γ, e) given S, the equation $\beta(z) = S(z)$ implicitly defines the analytic curve Γ. As a curve (near $\Gamma_\circ$) may thus be recovered from its Schwarz function (relative to $\Gamma_\circ$), we will speak informally of "the curve $S(z)$" when making local arguments.

Next, we note that if $S_1(z)$ and $S_2(z)$ are two such curves, the antiholomorphic reflection of S_1 in S_2 is given by *multiplication of Schwarz functions*:

$$S_3 = S_2 \cdot S_1 = S_2 \circ S_1^{-1} \circ S_2 \tag{2}$$

Also, left multiplication by P, $Q \mapsto l_P Q = P \cdot Q$ satisfies the formal properties defining a *symmetric space multiplication* in the sense of [Loos]: for each $P \in \Lambda$, $l_P : \Lambda \to \Lambda$ is an involutive automorphism of Λ with isolated fixed point P. It will be seen below that in fact l_P is the *geodesic symmetry with respect to* P in the standard symmetric space sense. The automorphism $Q \mapsto l_{Id} Q = Q^{-1}$ corresponding to reversal of geodesics through the base curve will appear more concretely, below.

The *canonical left action* $\lambda : G \times G_\sigma \to G_\sigma$ is given by the automorphisms $p \mapsto \lambda_g(p) = \sigma(g)pg^{-1}$. In the present context, if S is the Schwarz function of Γ and $g(z)$ is a conformal map defined near Γ then the Schwarz function of the image curve $g(\Gamma)$ is given by

$$\lambda_g(S) = \sigma(g) \circ S \circ g^{-1} \tag{3}$$

and $\lambda_g(S_2 \cdot S_1) = \lambda_g(S_2) \cdot \lambda_g(S_1)$ holds. The local action is transitive on simple closed analytic curves, as a conformal map $g(z)$ between interiors of such curves extends conformally to the boundary. We will be especially interested in $g = h$ belonging to the diffeomorphism group H of the base curve, in which case $\sigma(h) = h$ and the action reduces to ordinary conjugation. Determining the orbits of this restricted action is far from trivial.

Change of base curve rules require more explicit notation. E.g., if P, Q, R are three analytic curves and S_{PQ}, S_{PR} are the Schwarz functions of Q and R relative to P, and S_{QR} is the Schwarz function of R relative to Q, then

$$S_{PR} = S_{PQ} \circ S_{QR} \tag{4}$$

In particular, $S_{PQ} \circ S_{QP} = S_{PP} = Id$. Also, Equations 3, 4 give

$$S_{g(P)g(Q)} = g \circ S_{PQ} \circ g^{-1} \tag{5}$$

for $g(z)$ a conformal map defined near P, Q.

Concrete computations are simplest with base curve $\Gamma_\circ = \mathbb{R}$. Even for $\Gamma_\circ = S^1$, it pays to take advantage of the above rules rather than compute directly. For instance, setting $P = \hat{\mathbb{R}}, Q = S^1$ in Equation 4, we have $S_{PQ}(z) = S_{QP}(z) = 1/z$, and the Schwarz functions of a third curve R relative to P, Q are reciprocals: $S_{QR}(z) = 1/S_{PR}(z)$. Similarly, direct verification of the following proposition is easiest in the case $\Gamma_\circ = \mathbb{R}$, from which the general case follows by an application of Equation 5:

Proposition 1. *For a fixed base curve $\Gamma_\circ$, consider a time-dependent, parametrized analytic curve $x \mapsto \gamma(t, x)$, $x \in \Gamma_\circ$, and corresponding Schwarz function $S(t, z)$ relative to $\Gamma_\circ$. Letting $n(t, x)$ denote a unit normal vectorfield along the curve $\Gamma_t = \gamma(t, \Gamma_\circ)$, the normal variation field $\dot\gamma^n = \langle n, \dot\gamma \rangle n$ of γ is given in terms of t and z-derivatives of S by:*

$$\dot\gamma^n(t, x) = -\frac{1}{2}\dot S(t, \gamma(t, x))/S'(t, \gamma(t, x)) \tag{6}$$

Thus, if $\gamma(0, x) = x$, the initial variation of $S(t, z)$ may be regarded as a normal vectorfield along $\Gamma_\circ$: $\dot S(0, x) = -2\dot\gamma^n(0, x)$, $x \in \Gamma_\circ$.

Note: here we are using the identification $a + ib \leftrightarrow (a, b) \in T_{(x_0,y_0)}\mathbb{R}^2$; we will continue to do so thoughout §3, after which we will adopt complex vectorfield notation $w(z)\frac{\partial}{\partial z}$ and related formalism.

3. Geodesics in Λ

In differential geometry the notion of *geodesic* may be regarded as one of the most basic consequences of geometric structure. This is especially true in Riemannian geometry, where a geodesic may be described, locally, as the *shortest path* joining a given pair of points $p, q \in M$. However, other geometric settings do not allow geodesics to be characterized quite so simply. For instance, in the pseudo-Riemannian case an *indefinite* form $\langle X, Y \rangle$ on each tangent space T_pM defines the geometry and consequently "shortest paths" need not exist. Actually, one may continue to take a variational approach to geodesics in this case by invoking the more general *least action principle* (as it applies in physics): the integral $\int_a^b \langle \frac{d\alpha}{dt}, \frac{d\alpha}{dt} \rangle dt$, defined on a space of paths $\alpha : [a, b] \to M$ joining given nearby points $p = \alpha(a)$ and $q = \alpha(b)$, is required to be *stationary* (*critical*), but is not necessarily *minimized*.

Indeed, the latter case arises in our context, for it was seen in [C-L] that the space of circles in the Riemann sphere may be regarded as a three-dimensional symmetric subspace Λ^3 of the space of analytic curves Λ and that this geometry on Λ^3 is pseudo-Riemannian. The description of geodesics in Λ^3 consequently breaks into *spacelike, lightlike* and *timelike* cases, respectively, according to whether the velocity vector $X = \frac{d\alpha}{dt}$ satisfies $\langle X, X \rangle > 0$, $\langle X, X \rangle = 0$, or $\langle X, X \rangle < 0$. On the more concrete level, the corresponding family of "continuously reflected" circles has *two, one* or *zero* common points of intersection.

The discussion of Λ^3 in [C-L] led naturally to the question of whether or not the geometry of the whole space Λ likewise admitted a metric structure of some type. A negative answer is given in 6, based on the symmetries of Λ already described, above. Such a result is hardly surprising; one may want to bear in mind some of the many important *non-metric* geometries, where the underlying symmetries include, say, *dilations* (as in "school geometry"), *projective*, or *conformal* transformations. We will still be able to regard geodesics as basic objects in our geometry, where the intuition of "shortest path" should perhaps be replaced by "straightest path", as measured via the notions of *parallel transport* or *connection*, which we now proceed to introduce in our context.

According to the general theory of [Loos], a symmetric space multiplication directly induces an affine connection ∇, *the canonical connection.* In the present situation, this connection may be expressed as $\nabla_X Y = \overline{\nabla}_X Y - \frac{1}{2S_z}(X_z Y + Y_z X)$ using the local identification of Λ with Schwarz functions relative to a fixed base curve $\Gamma_\circ$. We will not require this formula—it is derived in [C-L] using $\Gamma_\circ = \mathbb{R}$—but we note that it inherits invariance properties (with respect to change of base curve and the canonical left action λ) from those of the multiplication rule Equation 2.

As a consequence one obtains an invariant *geodesic equation*:

$$0 = \nabla_{\dot{S}} \dot{S} = \ddot{S} - \frac{\dot{S}}{S'} \dot{S}' \tag{7}$$

In [C-L] this equation is also interpreted as a continuous limit of *iterated Schwarzian reflection*, obtained by allowing an initial pair of curves to approach each other. For the sake of "physical intuition", it should be noted also that such a geodesic equation is second order in time (as usual) and in this sense resembles *Newton's Law*: $F = ma$—even if it may not be a consequence of a least action principle!

As the above PDE may be written $\partial_t(\dot{S}/S') = 0$, a solution $S(t, z)$ must satisfy a first order linear PDE:

$$\frac{\dot{S}(t, z)}{S'(t, z)} = -2w(z) \tag{8}$$

Here, $w(z)$ is taken to be holomorphic on a neighborhood of $\Gamma_\circ$. If we change base curve to Γ_1 and write $S_1 = S_{\Gamma_1\Gamma_\circ}$, then the new Schwarz function $\tilde{S}(t,z) = S_1 \circ S(t,z)$ satisfies again $\dot{\tilde{S}}(t,z)/\tilde{S}'(t,z) = -2w(z)$. (As a special case of this invariance property, the base curves $\Gamma_\circ = \mathbb{R}$ and $\Gamma_\circ = S^1$ yield reciprocal pairs of solutions $S, 1/S$.) On the other hand, if we leave the base curve alone but conformally transform the evolving curve by $g(z)$, we find that $\tilde{S} = \lambda_g(S) = \sigma(g) \circ S \circ g^{-1}$ satisfies $\dot{\tilde{S}}(t,z)/\tilde{S}'(t,z) = -2\tilde{w}(z)$, where $\tilde{w}$ is given by the transformation rule

$$\tilde{w}(z) = \lambda_{g*}w(z) = g'(g^{-1}(z))w(g^{-1}(z)), \tag{9}$$

and Equation 7 continues to hold.

Equation 8 is written with the factor -2 so that, according to Proposition 1, $w(\gamma(t,x))$ describes the *normal velocity field* of the corresponding moving curve $\Gamma_t = \gamma(t, \Gamma_\circ)$:

$$\dot{\gamma}^n(t,x) = w(\gamma(t,x)), \;\; x \in \Gamma_\circ \tag{10}$$

Remark 1. Comparing Equations 6, 8 and 10 one arrives at the following important conclusion: What distinguishes geodesic motions among the more general variations representable via time-dependent Schwarz functions is that the former are generated by *time-independent* holomorphic vectorfields!

This interpretation may be realized more concretely. First observe that if we fix the initial condition $S(0,z) = z$, Equation 8 is but a differentiated version of the equation $S(t+u,z) = S(t, S(u,z))$ describing a one-parameter group $S(t,z)$. In particular, a time-dependent Schwarz function representing a geodesic Γ_t departing from $\Gamma_\circ$ at $t = 0$ satisfies $S(-t,z) = S^{-1}(t,z) = \sigma(S)(t,z)$ (and the automorphism $S \mapsto \sigma(S)$ clearly reverses geodesics through $\Gamma_\circ$ as claimed earlier). Now let $\theta \mapsto x(\theta) \in \Gamma_\circ$ be a parametrization of $\Gamma_\circ$ with real parameter θ, and set $\zeta(t,x) = S(-t/2, x)$ for $x = x(\theta) \in \Gamma_\circ$. Using the group law one obtains: $S(t, \zeta(t,x)) = S(t/2, x) = \sigma(S)(-t/2, x) = \beta(S(-t/2,x)) = \beta(\zeta(t,x))$—precisely the condition for $\zeta(t,x)$ to lie on Γ_t. In fact, $\zeta(t, x(\theta))$ parametrizes Γ_t by *normal motion*; for Equation 8 implies $\dot{\zeta}(t,x) = \zeta'(t,x)w(x)$, hence the ratio $\partial_t\zeta/\partial_\theta\zeta = w/x'$ is imaginary—being a ratio of normal and tangent vectors along $\Gamma_\circ$. It follows that $\dot{\zeta}^n(t,x) = \dot{\zeta}(t,x)$. To summarize, we have the following

Proposition 2. *Let Γ_t be a geodesic emanating from $\Gamma_0 = \Gamma_\circ$, with Schwarz function $S(t,z)$ satisfying Equation 8 and $S(0,z) = z$. Then Γ_t has* **canonical parametrization**

$$\zeta(t,x) = S(-t/2, x), \;\; x \in \Gamma_\circ \tag{11}$$

describing normal motion of $\Gamma_t = \zeta(t, \Gamma_\circ)$. Further, $t \mapsto \zeta(t,x)$ satisfies the following ODE initial value problem with parameter $x \in \Gamma_\circ$:

$$\dot{\zeta}(t,x) = w(\zeta(t,x)), \;\; \zeta(0,x) = x, \tag{12}$$

We will also use the notation $Exp = Exp_\circ$ for the *exponential map based at* $\Gamma_\circ$; namely, $\Gamma_t = \zeta(t, \Gamma_\circ) = Exp(tw)$ is the unique geodesic with *initial point* $\Gamma_\circ$ and *initial tangent* $w(z)$. Of course, $Exp(tw)$ is also given by the time-dependent Schwarz function $S(t,z) = \zeta(-2t, z)$—the solution to the parametrized ODE: $\dot{S}(t,z) = -2w(S(t,z)), \;\; S(0,z) = z$. Here we have replaced $x \in \Gamma_\circ$ by z in a neighborhood U of $\Gamma_\circ$ where S is holomorphic (say, in view of analytic dependence of solutions on parameters). Passing from Equation 8 to the latter ODE corresponds to the *method of characteristics.*

4. Holomorphic flows and trajectories of differentials

In the previous section it was notationally expedient to write $w(z)$ in place of $W = w(z)\partial_z$—essentially using local coordinates to identify a holomorphic function with a holomorphic vectorfield. Presently, the operator notation $W = w(z)\partial_z$ becomes much more appropriate, as we consider duality and other invariant notions of geometric and Lie-algebraic structure. Though the complex vectorfield formalism is perhaps unintuitive, it also helps distinguish the planar flows we wish to consider from the *ideal fluid flows* which have been associated commonly with holomorphic functions since the nineteenth century, when Riemann and others considered physical applications and interpretations of function theory. (The difference between the two types of planar flows will be mentioned below, and [C-L] includes a fuller explanation.)

To begin, we identify $W = \frac{1}{2}(u+iv)(\partial_x - i\partial_y)$ as needed with its *real part* $X = Re(W) = u\partial_x + v\partial_y$; the *imaginary part* $Y = Im(W) = -v\partial_x + u\partial_y$ may be computed as $Y = JX$, hence, one may always recover $W = \frac{1}{2}(X - iY)$. Thus, e.g., we will associate to W the *real* planar flow defined by X and its *conjugate flow* given by Y. The flow generated by W with *complex time* $\tau = t + is$ incorporates both of these. (Our terminology and notation here mostly follows that of [Mucino-Raymundo]; see also [Kobayashi-Wu], pp. 70-71, for standard identifications in the holomorphic vectorfield context.)

Via duality $W = w(z)\partial_z \leftrightarrow \omega = dz/w(z)$, one may describe the trajectories of X (and those of Y) in terms of the corresponding differential $\omega = \dfrac{u - iv}{u^2 + v^2}(dx + idy)$ or quadratic differential $p = \omega^2 = dz^2/w^2$: the *horizontal* (*vertical*) *trajectories* satisfy the equation $0 = Im(\omega) = \dfrac{udy - vdx}{u^2 + v^2}$, i.e., $p > 0$ ($0 = Re(\omega) = \dfrac{udx + vdy}{u^2 + v^2}$, i.e., $p < 0$). We note that the corresponding (*incompressible, irrotational*) "fluid flows" $X/|X|^2, Y/|Y|^2$ determine the same orthogonal pair of singular foliations (*streamlines* and *equipotentials*). Unlike the latter fields, however, X and Y commute, as a consequence of the Cauchy-Riemann equations for u, v. By the same token, ω is closed, with local potential $\phi(z) = U(x,y) + iV(x,y) = \int_{z_0}^{z} \omega$ defining a holomorphic coordinate (*natural parameter*) $\tau = t + is = \phi(z)$ near z_0, a regular point. ϕ maps small "curvilinear rectangles" bounded (and foliated) by U and V level sets onto Cartesian rectangles $t_1 < t < t_2,\ s_1 < s < s_2$, and gives local canonical forms $W = \partial_\tau,\ \omega = d\tau,\ p = d\tau^2$.

Returning to geodesics $\Gamma_t = \zeta(t, \gamma_\circ) = Exp(tW)$ in Λ, the velocity field of the canonical parametrization ζ might now be properly written $X = Re(W)$—in place of w in Equation 12—but we will simply display $W = w\partial_z$ in our examples, below. We compute $\zeta(\tau, z)$ by solving:

$$\omega = d\zeta/w(\zeta) = d\tau,\ \zeta(0, z) = z \tag{13}$$

In terms of local potential $\phi(z)$ as above, we have the implicit solution

$$\phi(\zeta(\tau, z)) = \phi(z) + \tau, \tag{14}$$

valid in the neighborhood of a regular point z_0, for small τ. We may subsequently specialize to real time $\tau = t$ and $z = x \in \Gamma_\circ$ to obtain the canonical parametrization $\zeta(t, x)$. For fixed t_0, the curve $\Gamma_{t_0} = \zeta(t_0, \Gamma_\circ)$ belongs locally to a vertical trajectory of $p = \omega^2$, and satisfies $U = U_0 = constant$. In fact, using $\tau = t$ and $z = x \in \Gamma_\circ$, the real part of Equation 14 becomes $U(\zeta(t, x)) = U(x) + t = U_0 + t$ (note Y is tangent to Γ_0, and $dU(Y) = Re(\omega)(Re(-iW)) = 0$). On the other hand, a point starting at $x_0 \in \Gamma_\circ$ follows a horizontal trajectory $\zeta(t, x_0)$ satisfying $V = const$. The *conjugate geodesic* to Γ_t, denoted $\star\Gamma_s = \zeta(is, \star\Gamma_\circ)$, is comprised of the latter curves and may be associated with $\star\omega = -i\omega$ and JW; it is determined locally by Γ_t and initial orthogonal curve $\star\Gamma_\circ$. With respect to this curve, $\star\Gamma_s$ has Schwarz function $\star S(s, z) = S(is, z)$.

In simple examples, such computations will be seen to have more or less global interpretations—even in the presence of singular points—by due consideration of multivaluedness, etc. For general purposes, however, the above description is inadequate, and one requires a suitable theory of canonical forms. There is such a local canonical form result for a meromorphic quadratic differential $p = P(z)dz^2$ near a singular point (see [Strebel]); but we do not invoke such results from the standard theory, as our context is rather special. For one thing, it will suffice for our purposes to restrict to *orientable* quadratic differentials $p = \omega^2$ (see, however, Remark 3). Further, we are primarily concerned with *non-vanishing* meromorphic differentials ω defined (and *real* or *imaginary*) along $\Gamma_\circ$, as we are currently assuming W holomorphic (and *tangential* or *normal*) along $\Gamma_\circ$. Nor is our situation covered as a special case of the above-mentioned local normal form, as we will require a description of ω near $\Gamma_\circ$—not merely in the vicinity of a single point $z_0 \in \Gamma_\circ$. In §10, Theorem 2, we give the relevant statement (and direct proof).

It will nevertheless be useful to recall one of the basic ideas in the standard theory; namely, issues of (global as well as local) equivalence for quadratic differentials may be discussed in terms of isometric equivalence with respect to the (singular) Riemannian metric $g = |P(z)|dzd\bar{z} = \dfrac{dx^2+dy^2}{u^2+v^2} = \phi^*(dx^2+dy^2)$. Near a regular point the commuting fields X, Y are orthonormal with respect to this flat metric, which is the pull-back of the Euclidean metric by the local isometry ϕ. The vectorfields X and Y generate local isometries and their trajectories are unit speed geodesics. The lengths of curves $\mathcal{L} = \int_\gamma |P|^{1/2}|dz|$ and areas of regions $\mathcal{A} = \int\int_{\mathcal{R}} |P|dxdy$ are useful, e.g., in discussing, moduli of ring domains. Our concrete examples will include geometric descriptions along such lines, with everything expressed in terms of ω: $p = \omega^2$, $g = \omega\bar{\omega}$, $\mathcal{L} = \int_\gamma |\omega|$, etc.

Example 1. Circles: The unoriented circles in the Riemann sphere form a three-dimensional subspace $\Lambda^3 \subset \Lambda$ in which the multiplication reduces to the usual circle inversion. Λ^3 is (double-covered by) a *generalized sphere* in Lorentzian four-space, and the three isometry classes of geodesics in Λ^3 have *timelike, spacelike, or lightlike* initial velocity vectors, corresponding to $W = w(z)\partial_z$ having no zeroes, two zeros, or a double zero on $\Gamma_\circ = S^1$ (as discussed in [C-L] using $\Gamma_\circ = \mathbb{R}$).

Here we consider geodesics in Λ^3 to illustrate conformal invariants and flat geometries. It suffices to look at geodesics departing from the base curve $\Gamma_\circ = S^1$ with the following initial velocities:

$$\begin{aligned} W_- &= -i\partial_\theta = z\partial_z \\ W_+ &= -i\sin\theta\partial_\theta = \frac{1}{2i}(z^2-1)\partial_z \\ W_0 &= -i(1-\cos\theta)\partial_\theta = -(z-1)^2\partial_z \end{aligned}$$

(We note $W_-, W_+,\ W_0$ generate the algebra $\mathfrak{sl}(2,\mathbb{C})$, with Cartan-Killing form satisfying $sgn(\langle W_\sigma, W_\sigma\rangle) = \sigma$.)

a) $W_- \leftrightarrow \omega = d\log z$. With $\phi = \log z$, Equation 14 gives $\zeta(t, e^{i\theta}) = e^{t+i\theta}$. $\Gamma_t = C_{e^t}$ is the "expanding circles" foliation of $\mathbb{C}^\times = \mathbb{C}\setminus\{0\} : U = \ln|z| = t$. We note that Γ_t has Schwarz function $S(t,z) = e^{-2t}z$ (w.r.t. $\Gamma_\circ = S^1$). The conjugate geodesic $\star\Gamma_s$ has Schwarz function $\star S(s,z) = e^{-2is}z$ (w.r.t. $\star\Gamma_\circ = \mathbb{R}$), and gives the "rotating lines" foliation, whose restriction to $\mathbb{C}^\times$ is given by the pairs of opposite rays $V = \arg z = s + k\pi,\ k = 0, 1$.

The metric $g = \dfrac{dx^2+dy^2}{x^2+y^2}$ makes $\mathbb{C}^\times$ a standard infinite cylinder of circumference $\mathcal{L} = \int_{S^1}|\omega| = \int_{S^1} -i\omega = 2\pi$, with (identity component of the) isometry group generated by X, Y. Restriction to a symmetric time interval $|t| < \mu/2$ gives the finite cylinder $S^1 \times (-\mu/2, \mu/2)$ and corresponding annulus $A_r = \{z : 1/r < |z| < r\}$, $r = e^{\mu/2}$, which provides the *normal form* in our discussion of foliated ring domains, below. The modulus $\mu = \ln r^2$ of A_r is related to the geometry of the cylinder by: $\mu = 2\pi\mathcal{A}/\mathcal{L}^2 = h$. The self-explanatory formula reflects the general characterization of ring domain moduli as solutions to extremal problems (the *length-area method*); a related extremal problem will play a role in the next section.

b) In the second case, $\omega = 2idz/(z^2-1) = d\phi$, $\phi = i\log\frac{z-1}{z+1}$, and

$$\zeta(t, e^{i\theta}) = \frac{(e^{i\theta}+1) + e^{-it}(e^{i\theta}-1)}{(e^{i\theta}+1) - e^{-it}(e^{i\theta}-1)} = \frac{\cos\frac{1}{2}(\theta-t) + i\sin\frac{1}{2}(\theta+t)}{\cos\frac{1}{2}(\theta+t) - i\sin\frac{1}{2}(\theta-t)}.$$

The 2π-periodic geodesic Γ_t foliates $\hat{\mathbb{C}}\setminus\{-1,1\}$ by circles through $z=\pm1$, namely, the pairs of level sets $U(z)=-\arg\frac{z-1}{z+1}=t\mp\frac{\pi}{2}$; here the two signs $\mp$ correspond to the two circular arcs meeting at ±1. $\star\Gamma_s$ consists of the orthogonal circles $V(z)=\ln|\frac{z-1}{z+1}|=s$.

The points $\pm1\in S^1$ are *pivot points* of Γ_t—stationary points at which the tangent line to Γ_t rotates. In general, pivot points correspond to simple poles of ω, and the residue of ω at such z_0 determines the constant rate of rotation of Γ_t (as shown in [C-L]): $d\alpha/dt=-i/res_{z_0}\omega$. Thus, the rotation rates at $z_0=\pm1$ are $\frac{d\alpha}{dt}=\mp1$.

Up to conjugation by a Möbius transformation, cases a) and b) are essentially the same—but with roles of Γ_t and $\star\Gamma_s$ reversed. In particular, $\hat{\mathbb{C}}\setminus\{-1,1\}$ is isometric to the cylinder of part a). This is a good place to take note of a divergence between the *foliated flat geometry* and *geodesic interpretations*. On the one hand, there is no place for the points ±1 themselves in the cylindrical geometry $(\hat{\mathbb{C}}\setminus\{-1,1\},g)$. On the other hand, these pivot points belong to the corresponding geodesic Γ_t, interpreted as a continuous variation of S^1 given by ζ or the Schwarz function $S(t,z)=\dfrac{(z+1)+e^{2it}(z-1)}{(z+1)-e^{2it}(z-1)}$. The same reasoning requires *two* values $U(z)=t\mp\frac{\pi}{2}$ above ($V=\arg z=s+k\pi$ in a)), illustrating the fact that a given curve Γ_{t_0} need not belong to a single trajectory of p.

c) In the last case we have $\omega=\dfrac{-dz}{(z-1)^2}=d\phi$, $\phi=\dfrac{1}{z-1}$, $\zeta(t,e^{i\theta})=\dfrac{e^{i\theta}+t(e^{i\theta}-1)}{1+t(e^{i\theta}-1)}$, and Γ_t, $\star\Gamma_s$ foliate $\hat{\mathbb{C}}\setminus\{1\}$ by circles tangent to the line $x=1$ at $z=1$, $U(z)=Re(\dfrac{1}{z-1})=\dfrac{x-1}{(x-1)^2+y^2}=t-\dfrac{1}{2}$, and the orthogonal circles $V(z)=Im(\dfrac{1}{z-1})=\dfrac{-y}{(x-1)^2+y^2}=s$. The map $\phi:\hat{\mathbb{C}}\setminus\{1\}\to\mathbb{C}$ defines an isometry between $(\hat{\mathbb{C}}\setminus\{1\},g)$ and the Euclidean plane. The vanishing of the residue of ω at 1 implies that the *Kasner invariant* $\mathcal{K}$ associated with the pair of tangent curves $\Gamma_\circ$ and Γ_t is constant in time ($\mathcal{K}=0$ actually)—see [C-L]. Another exceptional feature of this case is the fact that the isometry group is three dimensional, generated by X,Y, and the rotation field $Z=Re(-i(z-1)\partial_z)$; of course, rotations do not preserve the horizontal foliation. This phenomenon is explained in more general Lie algebraic terms in Remark 7.

The following example provides a recipe for a large family of geodesics which are locally equivalent to the expanding circles C_{e^t}, and illustrates the computational value of some of the above quantities.

Example 2. Poles of ω as obstacles: One may design "obstacle courses" for geodesics by placing N pairs of poles $\alpha_n,1/\bar{\alpha}_n$ off the initial curve S^1. Consider $g(z)=\dfrac{Cz^N}{\prod_{n=1}^N(z-\alpha_n)(1-\bar{\alpha}_nz)}$, $C\in\mathbb{R}^\times$, a general rational function which is real on S^1 and nonvanishing away from $0,\infty$. Define the differential $\omega=g(z)dz/z$ and dual vectorfield $W=\frac{z}{g(z)}\partial z$. The flows of $X=Re[W],Y=Im[W]$ may be used to generate a figure consisting of curves $\Gamma_{j\Delta t}$, $-J\le j\le J$, spaced by uniform time increment Δt; each curve is a trajectory of X, with initial condition determined by the conjugate flow Y, starting at a convenient point on $\Gamma_\circ=S^1$. As W is holomorphic on $\mathbb{C}^\times$, Γ_t evolves smoothly—avoiding the zeros $\alpha_n,1/\bar{\alpha}_n$ of W—until Γ_t reaches the singularities $0,\infty$. An example with $N=8$ is shown in Figure 1 (indicated in the figure are the "obstacles": $\pm2\pm2i,\pm10,\pm10i$ and conjugate reciprocals).

Due to the singularities, the required computations are numerically sensitive. The following procedure yields good results. First, a residue calculation determines C so that $\mathcal{L}=\int_{S^1}|\omega|=2\pi$; this standarizes the parameter domain $[0,2\pi]$ for the flow Y (and makes $\Gamma_t\sim C_{e^t}$). Thus normalized, $\omega=d\phi$ is antidifferentiated to give a (multivalued) potential ϕ. Using the real part $U=Re(\phi)$, a potential difference $T=U(1)-U(0)=U(\infty)-U(1)$ is computed to determine the time it takes Γ_t to run into $0,\infty$. To display a bounded, but *near maximal* geodesic, one then makes choices

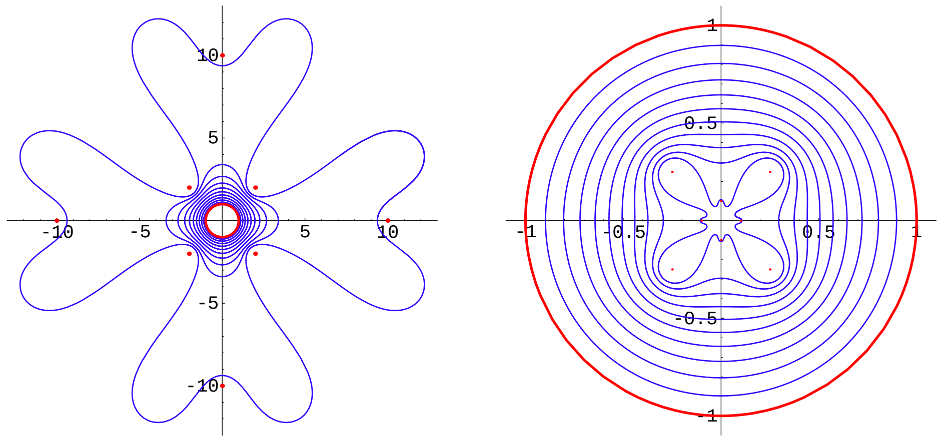

FIGURE 1. Poles of ω as obstacles for a geodesic Γ_t, shown for $t > 0$, *left*, and for $t < 0$, *right*.

to satisfy $J\Delta t = (1 - \epsilon)T$, for small $\epsilon > 0$. In the figure, $\epsilon = 1/10,000$; we note that the four "clover leaves" have scarcely begun to form until the last "instant"—this is the reason T must be computed precisely!

Applying contour plot methods directly to the real potential U might appear to be a simpler alternative to solving ODE's as above. We emphasize, however, that a given curve Γ_{t_0} does not quite coincide with a U-level set; in the present example, the latter may have "unwanted components" (even as level sets were "too small" in Example 1).

Example 3. String interactions: Selecting N pairs of simple poles $\alpha_n, 1/\bar{\alpha}_n$ and corresponding real residues $\pm C_n \in \mathbb{R}^\times$, we consider the differential $\omega = \dfrac{g(z)dz}{z} = \displaystyle\sum_{n=1}^{N} C_n(\frac{dz}{z-\alpha_n} - \frac{dz}{1-\bar{\alpha}_n z})$. As $t \to \pm\infty$, the asymptotic behavior of the corresponding flow $X = Re(W)$ near these poles resembles that of Example 1 a) near $0, \infty$; but ω also has $2N-2$ zeros, disrupting smooth evolution of Γ_t at intermediate times.

Here it is tempting to consider an alternative to the notion of geodesic as smoothly evolving Jordan curve Γ_t. Namely, $\omega = d\phi$ has single-valued real potential $U = Re(\phi)$, whose level sets $U = t$ define a solution Γ_t to our geodesic equation except at finitely many point-times (z_j, t_j) where topological transitions occur. (To compare with the *singular geodesics* considered in [C-L]: the topological transitions corresponded to cusps of Γ_t due to branch points of W and Γ_t was seen to be represented, nevertheless, by a *smooth* solution to a quartic, second order PDE.) Such a foliation by level sets $U = t$ is essentially the setting of "string diagrams" as considered, e.g., in [Krichever-Novikov], where an arbitrary compact Riemann surface is allowed in place of $\hat{\mathbb{C}}$.

Presently, the strings admit the following geometric description. Outside of a large enough compact subset $S \subset \hat{\mathbb{C}} \backslash \{\alpha_1, 1/\bar{\alpha}_1, \ldots \alpha_N, 1/\bar{\alpha}_N\}$ an individual "string" $s_n(t)$ looks like the standard cylinder of radius $|C_n|$. The "string interactions" are Euclidean surgeries among the cylinders taking place in S. In case $|\alpha_n| < 1$ and $C_n > 0$ for all n, we may assume ω has been normalized so that $C_1 + \ldots C_N = 1$ and consequently S^1 has length $\mathcal{L} = 2\pi$.

Figure 2 shows the interaction of two strings $s_1 = (1/2, 2)$ and $s_2 = (2/3, 3/2)$ with respective radii $C_1 = 5/17$ and $C_2 = 12/17$. Here one may recognize the pattern of *Cassinian curves*, symmetrized with respect to S^1 $(U = t = 0)$; approximations of the classical family appear inside $(t < 0)$ and outside $(t > 0)$ the unit circle. To obtain satisfactory output using the *Mathematica* command **ContourPlot**, one need only choose a contour increment ΔU so that the lemniscate level sets $U = \pm U(z_0)$ are included; here, z_0 is one of the two zeros of ω, $z_0 = (33 + \sqrt{305})/28 \approx 1.8$

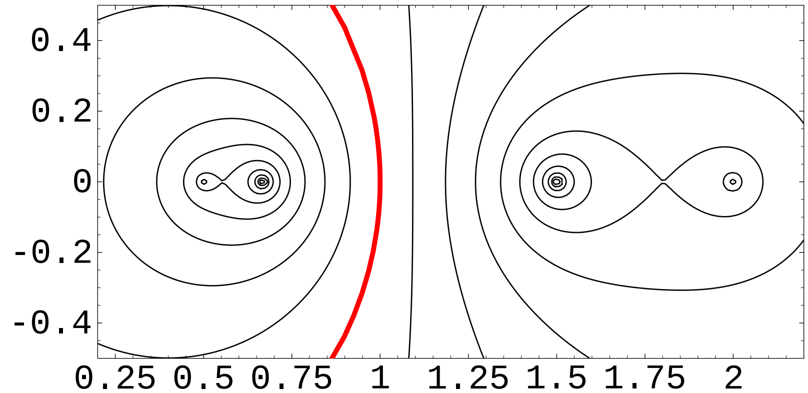

FIGURE 2. Two-string interaction.

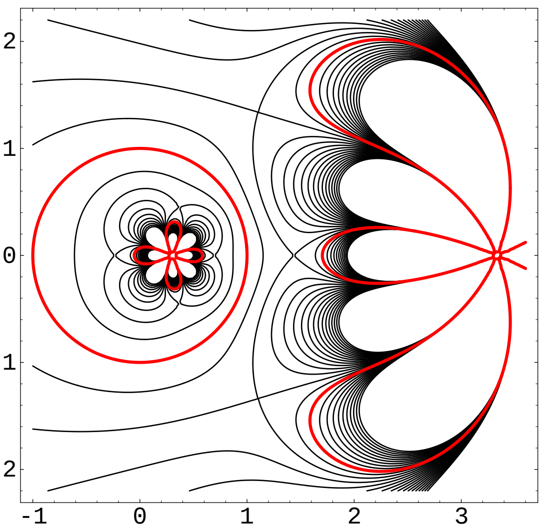

FIGURE 3. A singular initial curve.

(self-intersection point of the outside lemniscate). Finally, we note that by restricting time to $|t| < U(z_0)$, one may regard Γ_t as a *maximal geodesic* in the smooth sense.

Figure 3 shows a variation on the above construction, in which $\omega = d\phi$ has a single pair of *fifth* order poles $\alpha = 3/10,\ 1/\bar{\alpha} = 10/3$, along with four pairs of simple zeros $\beta_k,\ 1/\bar{\beta}_k$. The level sets $U = Re(\phi) = t$ describe "self-interactions of a single string" which is not asymptotically circular anymore, but a "flower" with *eight* $(= 2 \times 5 - 2)$ petals.

We wish to call attention to the more problematic *zero*-level set (highlighted), of which S^1 is a component. This figure illustrates but one of many possible ways the level set interpretation $\Gamma_t = U^{-1}(t)$ could substantially complicate even "short time" or infinitesimal description of geodesics. Instead, the present paper emphasizes smoothly evolving Jordan curves Γ_t, allowing us to adopt simple heuristic interpretations of $\Lambda,\ T_\circ\Lambda$, and clear-cut roles for the group $H = \mathrm{Diff}^\omega_+(S^1)$, the Witt algebra, etc. (In the future, we intend to explore interesting possibilities introduced by allowing variation fields, say, with poles on $\Gamma_\circ$, and further relations to [Krichever-Novikov], in which a variety of Lie algebras are associated with curves in Riemann surfaces.)

Remark 2. With the previous paragraph in mind, all subsequent plots will display geodesics Γ_t only up to first encounter with singularities; thus, we have opted for *clarity* at the cost of sometimes losing information. (Arguably, the resulting figures are also aesthetically inferior to those just considered!) In the sense that a "typical" geodesic runs into a natural boundary anyway—this is certainly the appropriate viewpoint for the next section—our blanket decision is further justified.

5. Foliation of singular ring domains by geodesics

For $1 < r < \infty$, let A_r be the S^1-*symmetric annulus* $A_r = \{z \in \mathbb{C} : 1/r < |z| < r\}$. As is well-known, any member of the family $\mathcal{A}$ of *ring domains* $A \subset \hat{\mathbb{C}}$ (topological open annuli with both boundary components consisting of more than one point) is conformally equivalent to A_r for exactly one value of r. The conformal class of A is thus described by its *symmetric radius* $r = r(A)$—but more standardly by its *modulus* $\mu = 2\ln r$ (the height of the equivalent right circular cylinder of radius *one*). Given one such conformal equivalence $g : A_r \to A$, any other has the form $g(e^{i\theta_0}z)$ or $g(e^{i\theta_0}/z)$. If A belongs to the family $\mathcal{A}_{\text{reg}} \subset \mathcal{A}$ of ring domains with boundary ∂A consisting of two simple closed analytic curves, such g extends conformally to ∂A_r.

Any $A \in \mathcal{A}$ has a unique *equator* (or *core curve*), the closed analytic curve $\Gamma_e \subset A$ well-defined by $\Gamma_e = g(S^1)$. There is an antiholomorphic involution of A which switches boundary components with fixed point set Γ_e; no other points may be fixed by such a map. The core curve is also the unique closed geodesic with respect to the complete hyperbolic metric on A of curvature -1 (see [McMullen], p. 12); in this metric, Γ_e has length $1/2\mu$. The "limiting case" $A_\infty = \hat{\mathbb{C}} \setminus \{0, \infty\}$ is not hyperbolic, but any circle $C_R = \{z : |z| = R\}$ may play the role of Γ_e.

The S^1-*symmetric* ring domains $\mathcal{A}^\circ = \{A : A = \beta(A)\}$ ($\beta(z) = 1/\bar{z}$) are precisely those with equator $\Gamma_e = S^1$. We may identify $\mathcal{A}^\circ_{\text{reg}} = \mathcal{A}_{\text{reg}} \cap \mathcal{A}^\circ$ with $\Lambda_+ \subset \Lambda$, the simple closed analytic curves in $\hat{\mathbb{C}}$ exterior to S^1; $\Gamma \in \Lambda_+$ determines $A_\Gamma \in \mathcal{A}^\circ_{\text{reg}}$ with boundary components Γ, $\beta(\Gamma) \in \Lambda_-$. We define the *radius* of $\Gamma \in \Lambda_+$ by $r(\Gamma) = r(A_\Gamma)$ and by $r(\Gamma) = 1/r(A_{\beta(\Gamma)})$ for $\Gamma \in \Lambda_-$. (We note r is the usual radius for circles C_r, but is *not* the *conformal radius* of Γ defined in [Kirillov], which is based on conformal mapping of discs rather than ring domains.)

Over the time interval $-\mu/2 < t < \mu/2$, our expanding circles geodesic $C_{e^t} = Exp(tz\partial_z)$ (Example 1) foliates A_r, $r = e^{\mu/2}$. A conformal image $A = g(A_r)$ is likewise foliated by $\Gamma_t = g(C_{e^t}) = Exp(t\lambda_{g*}z\partial_z)$ over the same time interval; the *duration* of Γ_t is the modulus of A. Conjugation of $e^{-2t}z$ by g gives the corresponding Schwarz function relative to the base curve $\Gamma_\circ = \Gamma_e = g(S^1)$.

In the symmetric case $A = g(A_r) \in \mathcal{A}^\circ$, g preserves $\Gamma_\circ = S^1$, and may therefore be regarded as the unique continuation of an analytic diffeomorphism $h = g|_{S^1} \in H = \text{Diff}^\omega_+(S^1)$—from now on we may write h in place of g. (That h is orientation-preserving on S^1 corresponds to $h(C_{e^t})$ moving "outward".) We thus have a local action of H on $\Lambda_\pm$ (or on $\mathcal{A}^\circ$), which is transitive on each level set Λ_r of the radius function $r : \Lambda_\pm \mapsto (0,1) \cup (0,\infty)$, by the conformal mapping facts quoted above. One also knows that the isotropy subgroup of a circle C_r consists of rotations about the origin; but one cannot identify Λ_r with H/S^1, as the action is only local.

On the other hand, the corresponding infinitesimal action is globally defined in the sense that, for any $h \in H$ and for any analytic vectorfield $W = w(z)\partial_z \in T_{\Gamma_\circ}\Lambda$ along S^1, we can compute $\lambda_{h*}W$ as in Equation 9. In particular, we may associate with any coset $hS^1 \subset H$ a *maximal geodesic* $\Gamma_t = Exp(t\lambda_{h*}z\partial_z)$, $-m/2 < t < m/2$, foliating a symmetric annular domain $A(h) = h(A_R)$, $R = R(h) = e^{m/2}$. Here, A_R is maximal with respect to conformal continuations of the form $h : A_r \to \hat{\mathbb{C}}$. We call $R = R(h)$ the *injectivity radius* of $h \in H$. On $A(h)$, $W = \lambda_{h*}z\partial_z$ is a nonvanishing holomorphic vectorfield whose real and imaginary parts are, respectively, "normal" and "tangent" to the boundary $\partial A(h)$ (as well as to S^1). Namely, $Im(W)$ generates the one-parameter group of automorphisms of $A(h)$ (which fixes the equator S^1), and the commuting fields $Re(W)$, $Im(W)$ are coordinate fields whose flows may be used to recover the conformal map $h : A_R \to A(h)$.

The ring domain $A(h)$ is necessarily *singular*: $A(h) \in \mathcal{A}^\circ_{sing} = \mathcal{A}^\circ \setminus \mathcal{A}^\circ_{reg}$. In the simplest case, $h(z) = (z-\alpha)/(1-\bar{\alpha}z)$ an automorphism of the unit disc, $A(h)$ misses just two points of $\hat{\mathbb{C}}$. The corresponding maximal geodesics Γ_t are easily shown to be the *only* ones with $R = \infty$. For a *finite geodesic* Γ_t, h might still continue analytically—but non-univalently—to the boundary C_R, with points of self-tangency $h(Re^{i\theta_1}) = h(Re^{i\theta_2})$ or non-regular points, $h(Re^{i\theta_0})$ with $h'(Re^{i\theta_0}) = 0$, at the *ends* $\Gamma_{m/2} = h(C_R), \Gamma_{-m/2} = h(C_{1/R}) \notin \Lambda_\pm$. (Such behavior is illustrated in the next example.) But in general, h does not extend analytically to (even part of) C_R, and $\Gamma_{\pm m/2}$ may exhibit all possible complexity of a natural boundary. (All the more curious is the implicit identification $\mathcal{A}^\circ_{sing} \simeq H/S^1$!)

Example 4. Extremal ring domain: We construct a geodesic from $Exp(tz\partial_z)$ as above by realizing the *symmetrized Teichmüller domain* $T_\rho = \hat{\mathbb{C}} \setminus \{[0, 1/\rho] \cup [\rho, \infty]\}$, $\rho > 1$ as the conformal image of a standard annulus A_R. The singular ring domain T_ρ is the solution to an extremal problem to be utilized below. The required conformal map h is given in terms of the Jacobi elliptic sine function $\mathrm{sn}(z,p)$ with *modulus* p, $0 < p < 1$ (or *parameter* $m = p^2$). We recall that $\mathrm{sn}(z,p)$ is an odd, doubly periodic, meromorphic function on $\mathbb{C}$; $\mathrm{sn}(z,p)$ has periods $4K$ and $2iK'$, where $K = K(p) = \int_0^1 dx/\sqrt{(1-x^2)(1-p^2x^2)}$ is the complete elliptic integral of the first kind and $K' = K'(p) = K(1-p^2)$ its complement; $\mathrm{sn}(z,p)$ has a simple zero at the origin, simple poles at $\pm iK'$, critical points at $\pm K$, and is otherwise free of such points on the closure of $\mathcal{R} = \{z = x+iy : -K < x < K,\ -K' < y < K'\}$. Finally, $\mathrm{sn}(z,p)$ is univalent on $\mathcal{R}$ and maps the latter conformally onto the domain $S_p = \hat{\mathbb{C}} \setminus \{[-1/p, -1] \cup [1, 1/p]\}$.

$h : A_R \to T_\rho$ is now defined as follows. Letting $R = e^{\pi K/K'}$, we first map A_R onto $\mathcal{R}$ via $z \mapsto w = \frac{K'}{\pi}\mathrm{Log} z$. Then $\mathcal{R}$ is mapped as above onto S_p via $w \mapsto Z = sn(w,p)$. Finally, letting $\rho = \dfrac{1+p}{1-p}$, S_p is taken to T_ρ by $Z \mapsto \zeta = \dfrac{1+pZ}{1-pZ}$.

The canonical parametrization of the resulting maximal geodesic Γ_t is then given in terms of the elliptic modulus p by:

$$\begin{aligned} \zeta &= h(e^{t+i\theta}) = \frac{1+p\,\mathrm{sn}(\frac{K'}{\pi}(t+i\theta),p)}{1-p\,\mathrm{sn}(\frac{K'}{\pi}(t+i\theta),p)},\ -\pi \le \theta < \pi,\ -\frac{m}{2} < t < \frac{m}{2}, \\ m &= \mu(T_\rho) = 2\ln R = 2\pi K(p)/K'(p) \end{aligned}$$

Here, m is the modulus of the above ring domains A_R, S_p, T_ρ. Note that h extends analytically but not conformally to the boundary of A_R; in fact, $h'(e^{\pm\mu/2}) = h'(-e^{\pm\mu/2}) = 0$ and $h(e^{\pm\mu/2+i\theta}) = h(e^{\pm\mu/2-i\theta})$, $0 < \theta < \pi$. The maximal geodesic Γ_t for $p = 1/10$ is shown in Figure 4; Γ_t fills the ring domain $T_\rho = \hat{\mathbb{C}} \setminus \{[0, 9/11] \cup [11/9, \infty]\}$, which has modulus $m \approx 2.68$.

One may check that $\Gamma_t = Exp(tW)$, where W is holomorphically defined on T_ρ by

$$W = \lambda_{h*} z\partial_z = \frac{2K'}{(1+\rho)\pi}\sqrt{z(\rho-z)(\rho z-1)}\,\partial_z \tag{15}$$

In the equivalent expression $W = -\frac{2iK'}{(1+\rho)\pi}\sqrt{\rho^2+1-2\rho\cos\theta}\,\partial_\theta$, the *positive* square root is chosen for $\theta \in \mathbb{R}$, so W is outward along S^1.

Remark 3. Were it not for our present focus on maximal geodesics foliating singular ring domains $A \subset \hat{\mathbb{C}}$, it would be natural to regard the above Γ_t as *half* of a closed geodesic foliating the torus $T^2 = T^+_\rho \cup T^-_\rho$ obtained by gluing two copies of T_ρ together. We note that $p = (\frac{(1+\rho)\pi}{2K'})^2 dz^2/z(\rho-z)(\rho z-1)$ defines a non-orientable quadratic differential on $\hat{\mathbb{C}}$, which lifts to orientable $\hat{p}$ on T^2. The resulting geodesic on T^2 corresponds to concatenation of $Exp(tW)$ with the other branch $Exp(-tW)$, yielding a closed geodesic of period $2m$.

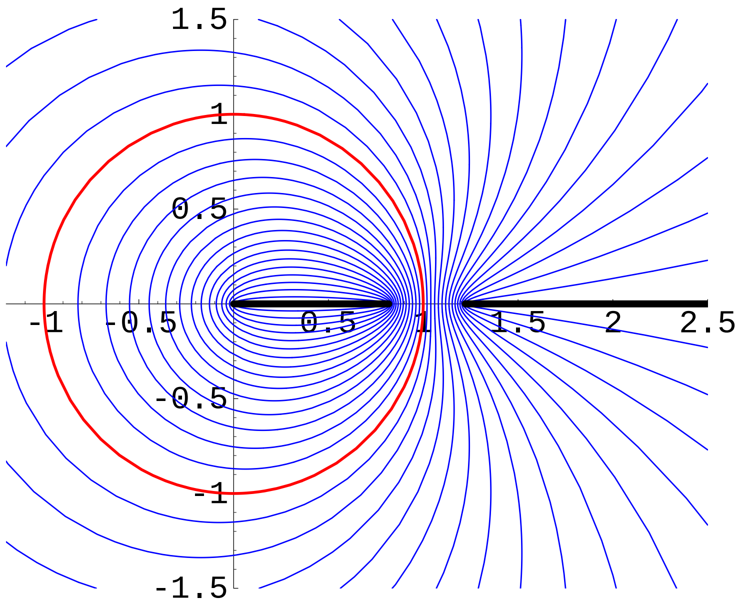

FIGURE 4. Maximal geodesic $\Gamma_t = Exp(tW)$ filling extremal ring domain $T_\rho = \hat{\mathbb{C}} \setminus \{[0, 9/11] \cup [11/9, \infty]\}$.

Next we characterize the orbit $H \cdot z\partial_z = \{W = \lambda_{h*} z\partial_z\}_{h \in H} \subset T_{\Gamma_\circ}\Lambda$ in terms of $W(e^{i\theta})$. First, $H \cdot z\partial_z$ lies in the open cone $\mathcal{C}_+ \subset T_{\Gamma_\circ}\Lambda$ of "outward fields" $W = z\nu(z)\partial_z = -i\nu(e^{i\theta})\partial_\theta$, where $\nu(e^{i\theta}) > 0$ for $e^{i\theta} \in S^1$. The dual differential $\omega = dz/w(z)$ of any $W = w(z)\partial_z \in \mathcal{C}_+$ is analytic along S^1 and gives a positive value for the *length* of S^1:

$$\mathcal{L}(\omega) = \int_{S^1} |\omega| = \int_{S^1} -i\omega = \int_0^{2\pi} d\theta/\nu(e^{i\theta}) \tag{16}$$

In particular, if ω is dual to $W \in H \cdot z\partial_z$, then $\mathcal{L}(\omega) = \mathcal{L}(dz/z) = 2\pi$. Conversely, if $W = w\partial_z \in \mathcal{C}_+$ satisfies $\mathcal{L}(dz/w) = 2\pi$ then $W \in H \cdot z\partial_z$. (This special case of Theorem 2 may be verified directly, using the natural parameters $\phi = \int_1^\zeta \omega$ and $\phi_0 = \int_1^z \frac{dz}{z}$ to construct the required conformal map $z \mapsto \zeta = h(z)$ on some annulus A_ϵ.)

Regarding the length condition $\mathcal{L}(\omega) = 2\pi$ as a kind of normalization, we will henceforth let $\hat{W} = \hat{w}(z)\partial_z$ denote an element of $\mathcal{C}_+$ satisfying $\mathcal{L}(dz/\hat{w}) = 2\pi$ (i.e., $\hat{W} \in H \cdot z\partial_z$); other vectors in $\mathcal{C}_\pm = \pm\mathcal{C}_+$ may then be written $W = \tau\hat{W}$, $\pm\tau > 0$. Further, such $\hat{W}$ determines a maximal geodesic $Exp(t\hat{W})$, $2|t| < m = m(\hat{W})$. The maximal subset of $\mathcal{C}_\pm$ on which the exponential map at $\Gamma_\circ$ is defined (in the above non-singular sense) is then given by:

$$U_\pm = \{W = \tau\hat{W} \in \mathcal{C}_\pm : 0 < |\tau| < m/2\}, \tag{17}$$

Starting with $\hat{W}$ as above, one would like to obtain information about m based on the locations of singularities of $\hat{W}$; note these must lie outside of $A(h)$ and occur in reflected pairs $\alpha, 1/\bar{\alpha} \in \hat{\mathbb{C}}$, by virtue of the fact that $\hat{W} \in T_{\Gamma_\circ}\Lambda$ satisfies $\hat{W}^* = -\hat{W}$. Thus, $\hat{W}$ has a symmetric *singular set*, $\mathcal{S} = 1/\overline{\mathcal{S}} \subset \hat{\mathbb{C}} \setminus h(A)$, consisting of those $\alpha \in \hat{\mathbb{C}}$ which do not lie in any connected neighborhood of S^1 to which $\hat{W}$ extends holomorphically and non-singularly. $\mathcal{S}$ contains at least two points $\alpha, 1/\bar{\alpha}$, which we may take to be $0, \infty \in \mathcal{S}$ by trivially modifying $\hat{W}$ by automorphism $\varphi_\alpha(z) = (z - \alpha)/(1 - \bar{\alpha}z)$. Henceforth, we will assume this has been done, and write simply $\hat{W}$ in place of $\lambda_{\varphi_\alpha *}\hat{W}$. We will give an upper bound on m in terms of the *regularity radius of* $\hat{W}$:

$$\rho = \rho(\hat{W}) = \sup\{r : \ A_r \cap \mathcal{S} = \emptyset\}. \tag{18}$$

(Of course, ρ depends on α; one may reasonably eliminate this dependence by taking the infimum over the original choices $\alpha \in \mathcal{S}(\hat{W})$.)

Proposition 3. *$Exp : U_\pm \to \Lambda_\pm$ is bijective, and maps $\tau\hat{W} \in U_\pm$ to a curve $\Gamma = Exp(\tau\hat{W})$ of radius $r(\Gamma) = e^\tau$. If $\hat{W}$ has singularities other than $0, \infty$ then $\hat{W}$ determines a finite maximal geodesic $Exp(t\hat{W})$, $2|t| < m(\hat{W})$, which foliates a singular ring domain $A \in \mathcal{A}_{sing}$. The duration $m(\hat{W}) = \mu(A)$ is bounded above in terms of the regularity radius $\rho = \rho(\hat{W})$ by:*

$$m \leq 2\ln R = \frac{2\pi K(p)}{K'(p)}, \qquad p = \frac{\rho - 1}{\rho + 1}$$

The bound is sharp, and is realized by the Teichmüller extremal domain $T_\rho = \hat{\mathbb{C}} \backslash \{[0, 1/\rho] \cup [\rho, \infty]\} = h(A_R)$, with h explicitly constructed via $\mathrm{sn}(z, p)$, the elliptic sine function of modulus p. It follows that $R(A) < 4\rho$ (4 being the best possible constant).

Proof. Surjectivity of Exp follows from conformal mapping as above. Namely, for $\Gamma \in \Lambda_+$, there exists $h : A_r \to A_\Gamma$ conformal up to the boundary ∂A_r, so $R(h) > r$. Letting $\hat{W} = \lambda_{h*} z\partial_z$ and $\tau = \ln r$, it follows that $\tau\hat{W} \in U_+$. Then $Exp(\tau\hat{W}) = Exp(\lambda_{h*}(\tau z\partial_z)) = \lambda_h(Exp(\tau z\partial_z)) = h(C_r) = \Gamma$, and $r(\Gamma) = r(C_r) = r$. Also, $Exp(-\tau\hat{W}) = \beta(\Gamma)$, so any $\Gamma \in \Lambda_\pm$ lies in the image of Exp.

To prove injectivity of Exp, suppose $Exp(t\hat{W}) = Exp(s\hat{V}) = \Gamma$, for some $t\hat{W}, s\hat{V} \in U_\pm$. Then $s = \mu(A_\Gamma)/2 = t$, and $\hat{W}, \hat{V}$ are nonvanishing, holomorphic vectorfields on A_Γ, with imaginary parts generating one-parameter groups of automorphisms of A_Γ. As $Aut(A_\Gamma)$ is one-dimensional, $\hat{W}$ and $\hat{V}$ must therefore be proportional; $\hat{W} = c\hat{V}$, in which only $c = 1$ is consistent with $\hat{W}, \hat{V}$ both being "normalized".

Finally, we show how to estimate μ. By rotation about the origin, we may assume the singular set of $\hat{W}$ includes the four points $\rho, 1/\rho, 0, \infty \in \mathcal{S}(\hat{W})$. Like the symmetrized Teichmüller domain T_ρ, the ring domain $A(h)$ lies in the complement of these four points and separates $\{0, 1/\rho\}$ from $\{\rho, \infty\}$. But the former is known to be *extremal*; that is, T_ρ has the largest modulus among ring domains satisfying this description. Up to Möbius equivalence, this is a special case of *Teichmüller's Module Theorem* —see Theorem 1.1 in [Lehto-Virtanen] (Ch.II, §1.2, p.55, where Teichmüller domains are defined by removing segments $[-r_1, 0]$ and $[r_2, \infty]$ and need to be S^1-symmetrized for our purposes). Thus, the given bound on μ follows from the computations in Example 4, and sharpness of the bound follows as well.

The function $f(p) = R(T_\rho)/\rho = \dfrac{1-p}{1+p} e^{\pi K(p)/K'(p)}$ increases monotonically from 1 to a limiting value of 4 over the interval $p \in [0, 1)$. The derivative of $f(p)$ is easily computed (using standard formulas for $\frac{dK}{dp}, \frac{dK'}{dp}$ and the Legendre relation): $f'(p) = \dfrac{f(p)}{2p(1-p^2)K'^2}(\pi^2 - 4pK'^2)$. Monotonicity thus reduces to the elementary estimate $pK'^2 < \pi^2/4$. The upper value $\lim_{p\to 1} f(p) = 4$ follows easily from Equation 112.04 in [Byrd-Friedman], where one may find also the other required facts about elliptic integrals. □

It is easy to see that no corresponding *lower* bound on the ratio $R/\rho = e^{\frac{m}{2}}/\rho$ exists, as illustrated in the first part of the following:

Example 5. Non-extremal singular ring domains:

a) Consider the differential $\omega = \dfrac{i}{a} e^{2\cos\theta} d\theta = \dfrac{e^{z+1/z} dz}{az}$, where $a = \frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta} d\theta$. The dual "outward" vectorfield $\hat{W}$ along S^1 lies in the orbit $H \cdot z\partial_z$, as it satisfies the normalization $\mathcal{L}(\omega) = 2\pi$; the geodesics $Exp(tz\partial_z)$ and $Exp(t\hat{W})$ are thus *locally* equivalent. As $\hat{W}$ has only the two essential singularities at $0, \infty$, it has infinite non-singularity radius $\rho(\hat{W}) = \infty$. Unlike $Exp(tz\partial_z)$, however,

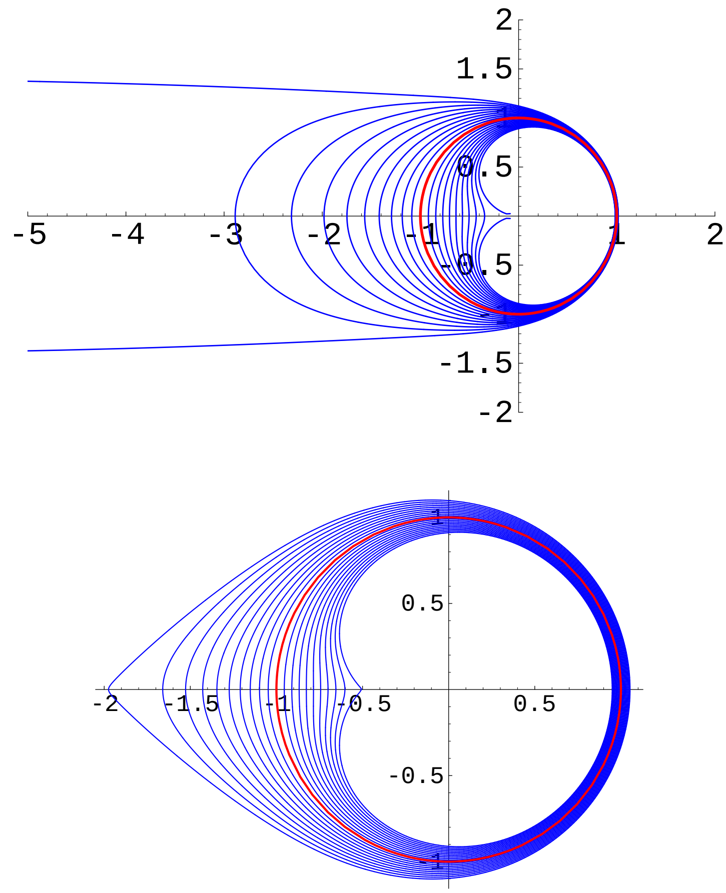

FIGURE 5. Maximal geodesics filling non-extremal singular ring domains.

$Exp(t\hat{W})$ runs into $\infty, 0$ in finite time $t = \pm m(\hat{W})/2$ as it fills in a singular ring domain of modulus $m(\hat{W}) \approx .1$ and symmetric radius $R = e^{\frac{m}{2}} \approx 1.05$. The latter domain is shown in Figure 5 (top).

b) Intermediate (and in some sense more typical) examples are provided by the family of differentials:

$$\omega_r = id\theta + \frac{2ir}{r^2+1}\cos\theta d\theta = \frac{dz}{z} + \frac{r}{r^2+1}(z+z^{-1})\frac{dz}{z}, \qquad 1 < r < \infty \tag{19}$$

(One may regard ω_r either as a deformation of the standard differential $\omega_\infty = dz/z$, or as a "truncated" version of part a) of this example.) ω_r has zeros at $-r$ and $-1/r$ and second order poles at $0, \infty$, so $\rho(\hat{W}_r) = r$. The maximal geodesic $\Gamma_t = Exp(t\hat{W}_r)$ foliates a singular ring domain A of modulus $\mu(A) = 2Re(\phi(-r)) = 2(\ln r + \frac{1-r^2}{1+r^2})$, where $\phi(z) = \int_1^z \omega_r = \log z + \frac{r}{r^2+1}(z - z^{-1})$. The ratio of $\mu(A)$ to the upper bound $2\pi K(p)/K'(p), p = (r-1)/(r+1)$ is small, for small r, but approaches 1 as $r \to \infty$.

The case $r = 2$ is shown in Figure 5 (bottom); in this case $\mu(A) \approx 0.19$, while $2\pi K/K' \approx 4$. Again, the geodesic Γ_t runs into singularities (at $-2, -1/2$) long before it has a chance to fill up much of the plane. In fact, in the right half-plane, A nearly coincides with the round annulus of modulus $\mu = .19$. As in a), the much larger left half of the annulus has little influence on μ. This reflects a basic principle of degenerating sequences of ring domains: if the boundary components approach each other while diameters of components themselves are bounded away from zero, the modulus tends to zero. (The quantitative expression of this principle is stated in terms of the spherical metric on $\mathbb{C}$—see Lemma 6.2 in [Lehto-Virtanen], Ch.I, §6.5, p.34).

Remark 4. In the remainder of the paper we consider the exponential map $Exp = Exp_\circ$ at S^1 for tangent fields $W = w(z)\partial_z$ which may have zeros along S^1. As above, $Exp(W)$ is defined for those $W \in T_\circ\Lambda$ for which the geodesic equation has a solution $t \mapsto \Gamma_t = Exp(tW) \in \Lambda$ for $t \in [0, 1]$. The zeros do not interfere with exponentiation *per se*; but the resulting geodesics have stationary points on S^1 and fail to sweep out planar domains—consider, e.g., $Exp(tW_+), Exp(tW_0)$ as in Example 1. Therefore one cannot directly apply standard conformal mapping theory to obtain global results as above. In particular, in place of the easily answered question of surjectivity of the restricted mapping $Exp : U_\pm \to \Lambda_\pm$ one confronts already the following hard

Problem: *Describe the domain $U \subset T_\circ\Lambda$ and range $V \subset \Lambda$ of Exp.*

The local version of this problem obtained by restricting to a neighborhood of a stationary point on S^1 is essentially the question of when two pairs of intersecting curves (*curvilinear angles* or *horn angles* in the case of curves meeting tangentially) are conformally equivalent; we refer the reader to [Davis] and references therein for an indication of the analytical subtleties inherent in this and related problems.

6. The infinitesimal symmetric space $T_\circ\Lambda$

To study the infinitesimal geometry of the space Λ of unparametrized analytic curves, we may fix a convenient base curve $\Gamma_\circ$ and identify nearby curves $\Gamma \in \Lambda$ with elements of $\mathcal{S}_\circ$, the space of Schwarz functions relative to $\Gamma_\circ$. As we presently wish to make use of the Lie algebra $\mathfrak{g}$ of analytic vectorfields defined along the circle, the natural choice is $\Gamma_\circ = S^1$ (even though $\Gamma_\circ = \hat{\mathbb{R}}$ is simpler for certain computations).

We consider G consisting of analytic diffeomorphisms defined near $\Gamma_\circ = S^1$ and the group $H = \mathrm{Diff}^\omega_+(S^1) \subset G$ of analytic diffeomorphisms preserving S^1 (and its orientation). We write $T_{Id}G \cong \mathfrak{g} \cong \mathfrak{h} \oplus i\mathfrak{h}$, expressing elements in either notation $A = a(\theta)\partial_\theta$ or $A = iz\mathbf{a}(z)\partial_z$. The case $A \in \mathfrak{h}$ corresponds to $\mathbf{a}(z)$ being real on S^1 (i.e., $a(\theta) = \mathbf{a}(e^{i\theta})$ is real for $\theta \in \mathbb{R}$), and A may be regarded as a real analytic tangent field along S^1. Likewise, $-iA = -ia(\theta)\partial_\theta = z\mathbf{a}(z)\partial_z$ denotes an element of $i\mathfrak{h}$, with the same reality condition on $\mathbf{a}(z)$, and may be interpreted as a normal vectorfield along S^1. In view of earlier discussion, we thus make the identifications $T_\circ\Lambda \cong T_{Id}\mathcal{S}_\circ \cong i\mathfrak{h}$.

$A \in \mathfrak{g}$ generates a local flow e^{tA} defined near S^1, and the formula

$$Ad_g A = \frac{d}{dt} g \circ e^{tA} \circ g^{-1}|_{t=0} = g' \circ g^{-1} A \circ g^{-1} = g' A|_{g^{-1}}, \tag{20}$$

defines the adjoint action $Ad : G \times \mathfrak{g} \to \mathfrak{g}$. The operator commutator of $A = iz\mathbf{a}(z)\partial_z, B = iz\mathbf{b}(z)\partial_z \in \mathfrak{g}$ is

$$[A, B] = -z^2(\mathbf{a}\mathbf{b}_z - \mathbf{a}_z\mathbf{b})\partial_z = i^2 z^2 \mathbf{a}^2 (\mathbf{b}/\mathbf{a})_z \partial_z = iz\mathbf{c}\partial_z \tag{21}$$

If $\mathbf{a}, \mathbf{b}$ are real on S^1 so is $\mathbf{c}$ —as is transparent in the other notation

$$[A, B] = [a\partial_\theta, b\partial_\theta] = (ab' - a'b)\partial_\theta = a^2(b/a)'\partial_\theta = c\partial_\theta \tag{22}$$

As usual, $ad : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, defined as the differential of $g \mapsto Ad_g A$ at $g = Id$ is computed in terms of the commutator:

$$ad_B A = \frac{d}{du} Ad_{e^{uB}} A|_{u=0} = [A, B]. \tag{23}$$

(The above notation fits the usual identification of the Lie algebra of a diffeomorphism group G with the *right-invariant* vectorfields on G—see [Milnor]—explaining the sign difference with the most standard matrix group convention $ad_A B = [A, B]$ based on left-invariant vectorfields.)

Either Equation 21 or 22 yields the *Witt algebra* bracket relations

$$d_n = -z^{n+1}\partial_z = ie^{in\theta}\partial_\theta, \qquad [d_m, d_n] = (m - n)d_{m+n}, \qquad m, n \in \mathbb{Z} \tag{24}$$

We will also use the basis

$$c_0 = \partial_\theta, \quad c_n = \cos(n\theta)\partial_\theta, \quad s_n = \sin(n\theta)\partial_\theta \quad n = 1, 2, 3, \ldots \tag{25}$$

for the real algebra $\mathfrak{h}$, though the bracket relations for $c_0 = -id_0$, $c_n = -\frac{i}{2}(d_n + d_{-n})$, and $s_n = -\frac{1}{2}(d_n - d_{-n})$ are not as convenient.

We return now to the geometry of Λ, and the heuristic $\Lambda \cong G/H$ (unparametrized curves are parametrized curves modulo reparametrizations). The corresponding *infinitesimal homogeneous space* (or *infinitesimal Klein geometry* as in [Sharpe]) is a geometry on the space $T_\circ\Lambda$ with symmetries defined by the *isotropy representation* $Ad^\Lambda : H \to GL(T_\circ\Lambda)$. In the *reductive* case—$\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{m}$, with $Ad_H(\mathfrak{m}) \subset \mathfrak{m}$—one may identify Ad^Λ with the "second factor" of the restriction to H of $Ad : G \times \mathfrak{g} \to \mathfrak{g}$ (see [Guest], p.16). The present case is especially simple, with $\mathfrak{m} = i\mathfrak{h}$ and Ad^Λ related to $Ad|_H$ by complex-linearity.

Comparison of Equations 9, 20 verifies that we have simply recovered the action previously considered, $Ad^\Lambda = \lambda_* : H \times T_\circ\Lambda \to T_\circ\Lambda$, placing the latter in a homogeneous space context. (Here we have restricted λ_* to H, incidentally, resulting in an actual group action on $T_\circ\Lambda$ by the Frechet Lie group H, etc.—but we will make no use of infinite-dimensional analysis here.) Finally, we mention that $\mathfrak{g} \cong \mathfrak{h} \oplus i\mathfrak{h}$ is the decomposition by ± 1-eigenspaces of the operation $W = \mathbf{w}(z)\partial_z \mapsto W^* = -z^2\mathbf{w}^*(z)\partial_z$ (dual to the operation on differentials $\omega \mapsto \omega^*$ discussed in §9). The involution * on $\mathfrak{g}$ yields the standard construction of the *Lie triple system* $[X, Y, Z] = [[X, Y], Z]$ on $\mathfrak{g}_- = i\mathfrak{h}$ (see [Loos]).

Equation 24 implies an important fact about the geometry of $T_\circ\Lambda$. Consider a bilinear form $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ which is invariant with respect to the adjoint action, that is, satisfying $\langle Ad_g\alpha, Ad_g\beta\rangle = \langle\alpha, \beta\rangle$ for all $\alpha, \beta \in \mathfrak{g}$, $g \in G$. Differentiation with respect to g at $g = Id$ gives the *ad*-invariance condition $\langle ad_\gamma\alpha, \beta\rangle + \langle\alpha, ad_\gamma\beta\rangle = 0$, for all $\alpha, \beta, \gamma \in \mathfrak{g}$. Expressed in the most symmetrical form, the condition is:

$$\langle[\alpha, \beta], \gamma\rangle = \langle\alpha, [\beta, \gamma]\rangle = \langle\beta, [\gamma, \alpha]\rangle, \quad \alpha, \beta, \gamma \in \mathfrak{g} \tag{26}$$

Take now $\alpha = d_n, \beta = d_m, \gamma = d_{-(n+m)}$, to get $(n - m)\langle d_{n+m}, d_{-(n+m)}\rangle = (2m + n)\langle d_n, d_{-n}\rangle = -(2n+m)\langle d_m, d_{-m}\rangle$. If $n = m \neq 0$, then $\langle d_n, d_{-n}\rangle = 0$. On the other hand, the choice $n = -m$ leads to the equation $2n\langle d_0, d_0\rangle = -n\langle d_n, d_{-n}\rangle = 0$, implying $\langle d_0, d_0\rangle = 0$ (using $n \neq 0$). Thus, as is well-known, there does not exist a non-degenerate bilinear form on $\mathfrak{g}$ which is invariant with respect to the adjoint action. (The related Remark 6 highlights simple differences between adjoint and coadjoint actions.) This conclusion is equivalent to corresponding statements about the restricted actions $Ad : H \times \mathfrak{h} \to \mathfrak{h}$ and $\lambda_* : H \times T_\circ\Lambda \to T_\circ\Lambda$, by virtue of complex extension.

Proposition 4. *There does not exist a (pseudo-Riemannian) metric on Λ with respect to which the canonical left action λ is isometric; in fact, no non-singular bilinear form on $T_\circ\Lambda$ is preserved by λ_*.*

As λ preserves the symmetric space multiplication defining the geometry of Λ—the covariant derivative, curvature, geodesics, etc.—the implication of the proposition is that Λ *is not a metric geometry.* In this sense, the symmetric subspace of circles Λ^3 considered in [C-L] has more rigid structure. Viewed intrinsically, $(\Lambda^3, \cdot)$ happens to be a *semisimple* symmetric space (see [Loos], Chapter 4); its Lorentzian metric and group of displacements $Isom_0(\Lambda^3, \rho) \simeq SL(2, \mathbb{C})$ may be recovered via the Ricci form $\rho(X, Y) = trace(Z \mapsto R(Y, Z)X)$ and Lie triple system of $(\Lambda^3, \cdot)$, respectively. (What other pseudo-Riemannian geometries—not just symmetric subspaces—are naturally embedded in Λ via our constructions is one of the general issues raised in [C-L] which we intend to address elsewhere.)

Another relevant feature of the Witt algebra is the fact that for each integer $n > 1$, there is an isomorphism $\Phi_n : \mathfrak{g} \to \mathfrak{g}_n$ mapping $\mathfrak{g} = \langle d_j\rangle$ onto a subalgebra $\mathfrak{g}_n = \langle D_j\rangle$, namely: $d_j \mapsto D_j =$

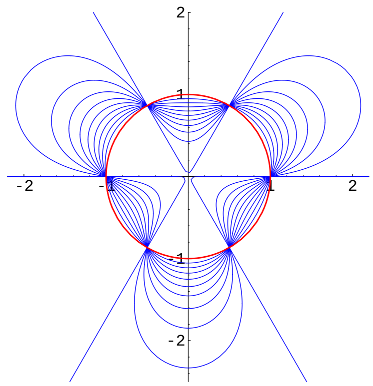

FIGURE 6. Geodesic with six pivot points.

$\Phi_n(d_j) = \frac{1}{n} d_{nj}$. The geometric significance of this fact is explained by the following heuristic argument. Letting $\varphi(z) = z^{1/n}$, $\varphi^{-1}(z) = z^n$, and allowing ourselves to make the exponent cancellation $(z^n)^{1/n} = z$, we compute $Ad_\varphi d_j = -Ad_\varphi z^{j+1}\partial_z = -\frac{1}{n}(z^n)^{(1-n)/n}(z^n)^{j+1}\partial_z = -\frac{1}{n}z^{1+nj}\partial_z = \frac{1}{n}d_{nj}$. In other words, the above isomorphism may be interpreted as the induced map $\Phi_n = \lambda_{\varphi *} = Ad_\varphi$ associated with the conformal equivalence between $\mathbb{C}^\times$ and its n-fold cover. While the global interpretation of this argument represents a basic challenge in the theory of Λ, its concrete significance is very simple: as the following example illustrates, φ turns "old geodesics" into "new" ones.

Example 6. Sinusoidal Variations: The geodesic $Exp(tW)$ may be computed explicitly in case iW is one of the Witt algebra basis elements c_n, s_n. It suffices to consider the following expressions:

$$\begin{aligned}
W_n &= \frac{1}{in}s_n = \frac{1}{in}\sin n\theta\partial_\theta = \frac{1}{2in}z(z^n - z^{-n})\partial_z \\
\omega_n &= \frac{2inz^{n-1}dz}{(z^n-1)(z^n+1)} = \frac{inz^{n-1}dz}{z^n-1} - \frac{inz^{n-1}dz}{z^n+1} \\
\phi_n &= i\log\frac{z^n-1}{z^n+1}; \qquad U = -\arg\frac{z^n-1}{z^n+1} = t - \frac{\pi}{2} \\
\zeta_n^n &= \frac{(e^{in\theta}+1) + e^{-it}(e^{in\theta}-1)}{(e^{in\theta}+1) - e^{-it}(e^{in\theta}-1)} = \frac{\cos\frac{1}{2}(n\theta - t) + i\sin\frac{1}{2}(n\theta+t)}{\cos\frac{1}{2}(n\theta+t) - i\sin\frac{1}{2}(n\theta - t)}
\end{aligned}$$

The case $W_1 = W_+$ was considered in Example 1. For $n \geq 2$, W_n is holomorphic on $\hat{\mathbb{C}} \setminus \{0, \infty\}$ and Γ_t is defined for $-\frac{\pi}{2} < t < \frac{\pi}{2}$. For such t, ζ_n is computed by taking the n^{th}-root of the above expression for ζ_n^n in such a way that $\theta \mapsto \zeta_n(t, e^{i\theta})$ is continuous; the result is a "sinusoidal" closed curve with a dihedral symmetry group of order $2n$ generated by $e^{i\pi/n}\beta(z)$, $\beta(z) = 1/\bar{z}$. In the context of the preceeding remarks, the interpretation $W_n = \lambda_{\varphi *}W_1$ is the infinitesimal version of the idea that conjugation of ζ_1 by $\varphi(z) = z^{1/n}$ yields $\lambda_\varphi\zeta_1 = \zeta_n$.

In the same vein, note that ω_n has simple poles at the $2n^{th}$ roots of unity $p_k = e^{ik\pi/n}$, with residue $res_{p_k}\omega_n = (-1)^k i$; by the remark in Example 1, the tangent line to Γ_t at the pivot point p_k therefore has constant angular rate of rotation $\frac{d\alpha}{dt} = -i/res_{p_k}\omega_n = (-1)^{k+1}$. Using $\zeta_n = \lambda_\varphi\zeta_1$, the fact that the rotation rates are independent of n may be regarded as an obvious consequence of conformal invariance.

Figure 6 shows Γ_t in the case $n = 3$, for positive times only. For $-\pi/2 < t < \pi/2$, each arch of Γ_t sweeps out a sector of $\hat{\mathbb{C}} \setminus \{0, \infty\}$, approaching perpendicularity to S^1 at its endpoints as $t \to \pm\frac{\pi}{2}$. (Were we to abandon regularity, Γ_t could be regarded as a π-period geodesic, passing repeatedly through the singular curve $\Gamma_{\pi/2}$ consisting of n straight lines intersecting at $0, \infty$.)

7. Geodesics generated by Rational Vectorfields

In this section we show how to represent all possible short-time behavior of geodesics—up to conformal equivalence near S^1—in the form $\Gamma_t = Exp(tW)$, where W is a rational vectorfield. Before presenting a comprehensive approach to this problem, we discuss some special constructions which are useful for generating further concrete examples. First we observe that the highly symmetrical geodesics $Exp(tW_n)$ in Example 6 may be *modulated* to give arbitrary rotation rates $d\alpha_k/dt = -i/res_{p_k}\omega$ (of alternating sign) at the pivot points $p_k \in S^1$. As $\omega_n = ind\theta/\sin n\theta$ has simple poles at the $2n^{th}$ roots of unity $p_k = e^{ik\pi/n}$, it suffices to multiply ω_n by a function $J(z)$ which is positive on S^1 with prescribed values $J(p_k) > 0$. In fact, suitable products $\tilde{\omega}_n = J\omega_n$ may be obtained within the class of rational differentials using methods of trigonometric interpolation. Namely, consider the *Fejér kernel* with index $\mu = 2n-1$:

$$\begin{aligned}\mathcal{K}(\theta) &= \frac{2}{\mu+1}\{\frac{\sin\frac{1}{2}(\mu+1)\theta}{2\sin\frac{1}{2}\theta}\}^2 = \frac{1}{4n}\frac{1-\cos 2n\theta}{1-\cos\theta} = K(e^{i\theta})\\ K(z) &= \frac{1}{4n}\frac{(z^{2n}-1)^2}{z^{2n-1}(z-1)^2} = \frac{1}{4n}z^{1-2n}\{\sum_{j=0}^{2n-1} z^j\}^2\end{aligned}$$

Note that $K(z)$ is analytic and real on S^1, with special values $K(1) = n$ and $K(p_k) = 0, k \neq 0$. Thus, a linear combination

$$\mathcal{J}(\theta) = \frac{1}{n}\sum_{j=0}^{2n-1}\lambda_j\mathcal{K}(\theta - j\pi/n), \tag{27}$$

satisfies $\mathcal{J}(k\pi/n) = \lambda_k$, and may be conveniently used to interpolate real periodic functions $f(\theta)$ at equally spaced points; using $\lambda_j = f(j\pi/n)$, the interpolating function $\mathcal{J}(\theta)$ is known as the *Jackson polynomial* of $f(\theta)$ (of index $\mu = 2n-1$).

For present purposes, the key advantage of Jackson interpolation (over, say, *Fourier-Lagrange* interpolation) is that $\mathcal{J}$ is uniformly bounded by the extreme interpolation values: $\min\{\lambda_j\} \leq \mathcal{J}(\theta) \leq \max\{\lambda_j\}$. (For a proof of this and for other relevant background, we refer the reader to [Zygmund], especially pp. 21-22.) In particular, if all interpolation values are positive, so is $\mathcal{J}(\theta)$. The rational vectorfield dual to $\tilde{\omega}_n = J\omega_n$ is thus analytic along S^1 and defines an element $\tilde{W}_n = W_n/J \in T_\circ\Lambda$; it has precisely the same zeros p_k along S^1 as W_n, but with magnitudes of the $2n$ rotation rates now fully controlled by the coefficients $\lambda_j > 0$.

Example 7. Modulated sinusoidal variation: We have already used the Jackson polynomial of index $\mu = 1$ (implicitly in Example 5), $\mathcal{J}(\theta) = 1 + \frac{2r}{(r^2+1)}\cos\theta = J(e^{i\theta})$, $J(z) = 1 + \frac{r(z^2+1)}{z(r^2+1)}, r > 1$. This time we consider the parameter range $-1 < r < 1$, for which the ratio $J(1)/J(-1) = (r+1/r-1)^2$ achieves all positive values. The product

$$\tilde{\omega}_1 = J\omega_1 = \omega_1 + \frac{2ir(z^2+1)dz}{(r^2+1)z(z^2-1)} = \frac{id\theta}{\sin\theta} + \frac{2ir}{(r^2+1)}\cot\theta d\theta \tag{28}$$

may therefore be scaled by $1/c > 0$ to give any desired negative and positive rotation rates $\frac{d\alpha}{dt} = \mp c/J(\pm 1)$ at the simple poles $z = \pm 1$, respectively. $\tilde{\omega}_1$ also has simple zeros at $-r, -1/r$, which eventually disrupt the smooth evolution of Γ_t. Setting $c = 1$, for convenience, the time it takes to reach $-1/r$ may be computed as the potential difference $\Delta U = U(-1/r) - U(i) = \frac{\pi}{2}\frac{(1-r)^2}{1+r^2}$, where U is the real part of the complex potential $\phi = i\log\frac{z-1}{z+1} + \frac{2ir}{r^2+1}\log\frac{z^2-1}{z}$. In the same time interval, the net rotation of Γ_t at $z = -1$ is $\Delta\theta = \Delta U/J(-1) = \pi/2$. This result is evident in Figure 7, which shows Γ_t for $r = 1/2$ on the positive time interval $0 \leq t < \Delta U = \pi/10$.

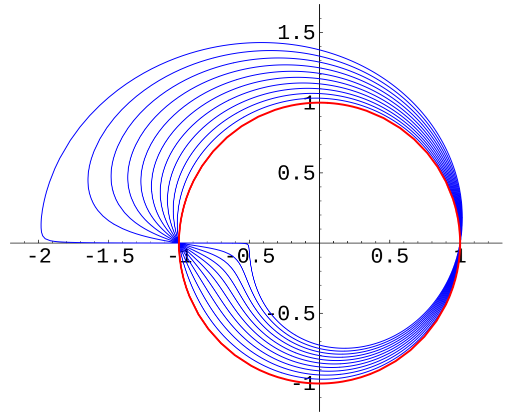

FIGURE 7. Geodesic with unequal pivot rates at $z = \pm 1$.

Remark 5. The length $\mathcal{L} = \int_{S^1} |\omega|$ for such a meromorphic differential $\tilde{\omega}_n = J\omega_n$ is infinite. On the other hand, the alternate expression $\mathcal{L} = \int_{S^1} -i\omega$ may still be assigned a finite value as a *Cauchy principal value integral* $\mathcal{L} = PV\{\int_{S^1} -i\omega\}$, obtained by passing to the limit $\epsilon \to 0$ in the integral of ω along S^1, omitting arcs $\theta_j - \epsilon < \theta < \theta_j + \epsilon$ about each pole $p_j = e^{i\theta_j}$. By this definition, $\mathcal{L}(\tilde{\omega}_n)$ is easily seen to vanish, by evenness of $\mathcal{K}(\theta)$ and oddness of ω_n in the variables $\theta_k = (\theta - k\pi/n)$; one could say the examples constructed thus far are *balanced* with respect to inward/outward variation from S^1. This non-genericity may be addressed as well using trigonometric interpolation. Specifically, one may multiply ω_n instead by an appropriate Jackson polynomial of index $\mu = 4n-1$, choosing the "new" coefficients so that all the negative extrema of $\dfrac{1}{\sin n\theta}$ get multipled by λ, say, and the positive extrema by 2λ. We omit details of the computation of $\mathcal{L}$ in terms of λ, showing that any $\mathcal{L} \in \mathbb{R}$ may be achieved by this method. The main point is that $\mathcal{L} = PV\{\int_{S^1} -i\omega\}$ equals the real part of the integral over a slightly smaller circle, $\mathcal{L} = Re(\int_{|z|=1-\epsilon} -i\omega)$ (as explained in §9), and may therefore be computed by residue calculus. Actually, the most important feature of the latter integral is that it is meaningful even when ω has higher order poles, as required below.

We turn now to the general problem of rational representation of analytic vectorfields $W \in T_\circ\Lambda$ up to conformal equivalence near S^1. While the latter are *normal* along S^1, it suffices (by $W \leftrightarrow iW$) to represent equivalence classes of *tangent* fields, i.e., orbits $Ad_H W$. Further, duality $(W = w(\theta)\partial_\theta \leftrightarrow \omega = d\theta/w(\theta))$ reduces the problem to achieving all possible meromorphic data along S^1 for *non-vanishing, real, meromorphic differentials on* S^1. Namely, using rational differentials of this kind, we need to realize prescribed pole orders $\mu_j \in \mathbb{Z}^-$ and residues $\nu_j \in \mathbb{R}$, along with given S^1-*period* $\mathcal{I} = Re(\int_{\gamma_\epsilon} \omega)$ (corresponding to the integral $\mathcal{L}$ of the above remark). Finally, we need to match specified *polarity* for the j^{th} pole, $\sigma_j = \pm 1 = sgn\ w(\theta)(\theta - \theta_j)^{\mu_j}$, θ near θ_j. This "sign" invariant enables us to conveniently express the *polarity constraints* Equation 34 restricting the above set of data—note, e.g., residues of consecutive simple poles must have opposite sign. For reading the following proof of Theorem 1, one may refer as needed to §§9,10 for fuller discussion of above data and the relevant classification result, Theorem 2.

Theorem 1. *For* $W = w(z)\frac{\partial}{\partial z} \in T_{\Gamma_\circ}\Lambda$*, the geodesic departing from* $\Gamma_\circ$ *with velocity* W *is representable in the form* $Exp(tW) = h(Exp(tV))$*, for small* $|t|$*, with* $h \in H$ *and rational vectorfield* $V = \frac{a_m z^m + \cdots + a_0}{b_n z^n + \cdots + b_0}\frac{\partial}{\partial z}$ *which can be constructed explicitly using meromorphic data of* $\omega = \frac{dz}{w(z)}$*, the dual differential to* W*.*

Proof. To construct the required rational differential $\omega = w(\theta)d\theta = \mathbf{w}(z)dz/iz$, we begin by assigning pole positions $p_1 = e^{i\theta_1}, \ldots, p_m = e^{i\theta_m}$ in counterclockwise order around S^1, as follows.

Note that there are an even number, $2N$, of prescribed odd order poles, by Equation 34. Assuming $N > 0$, we space these poles evenly around the circle at angles $\theta_{j_1} = \pi/N, \ldots \theta_{j_{2N}} = 2\pi$. The exact locations of even order poles will play no role—only the counterclockwise order must be respected.

We construct ω as a sum $\omega = \omega_s + \omega_r$ of *singular* and *regular* parts. $\omega_s = w_s(\theta)d\theta$ will be a linear combination of the following basic differentials (and "θ-translates" of these):

$$\begin{aligned} \alpha &= \frac{1}{2}\cot\frac{\theta}{2}d\theta = \frac{(z+1)dz}{2(z-1)z} \\ \beta^{-(2k+1)} &= \csc\theta(\cot\frac{\theta}{2})^{2k}d\theta = 2(-1)^k\frac{(z+1)^{2k-1}dz}{(z-1)^{2k+1}},\ k = 1, 2, \ldots \\ \beta^{-2k} &= \frac{d\theta}{(1-\cos\theta)^k} = -i(-2)^k\frac{z^{k-1}dz}{(z-1)^{2k}},\ k = 1, 2, \ldots \end{aligned}$$

Note that α has a simple pole at $z = 1$ with residue $Res[\alpha, 1] = 1$; the other finite pole $z = 0$ has residue $Res[\alpha, 0] = -1/2$ and consequently $Re[\int_{\gamma_\epsilon} \alpha] = 0$. For each integer $\mu < -1$, β^μ has a pole at $z = 1$ of order $ord_1\beta^\mu = \mu$ and zero residue; as β^μ has are no other poles, $Re[\int_{\gamma_\epsilon} \beta^\mu] = 0$. The corresponding differentials with pole $p = e^{i\theta_j}$ are the pull-backs, $\alpha_j = \rho_j^*\alpha$, $\beta_j^\mu = \rho_j^*\beta^\mu$, by the rotation $\rho_j(z) = e^{-i\theta_j}z$.

Now consider the following sum:

$$\omega_s = \sum_{j=1}^{m} \sigma_j\beta_j^{\mu_j} + \nu_j\alpha_j \tag{29}$$

Observe that ω_s satisfies the data $\{\mu_j, \nu_j, \sigma_j\}$, but gives $\mathcal{I} = 0$. To achieve nonzero $\mathcal{I}$, we may add the regular term $\mathcal{I}d\theta/2\pi = \mathcal{I}dz/2\pi iz$. (In the regular case $\omega_s = 0$, we have simply $\omega = \mathcal{I}d\theta/2\pi$, $\mathcal{I} \neq 0$.) The sum $\omega_s + \mathcal{I}dz/2\pi iz$ has just one defect: it may vanish at points on S^1. To correct this, assume first the prescribed orders μ_j are not all even—so $N \geq 1$. Then either the regular differential $\sin N\theta d\theta$ or its opposite agrees in sign with ω_s in the vicinity of every pole. It follows that for $\lambda \in \mathbb{R}$ of correct sign and sufficiently large magnitude, the sum $\omega_s + \mathcal{I}dz/2\pi iz + \lambda(z^{N-1} - z^{-N-1})dz$ is nonvanishing on S^1. As $\sin N\theta d\theta$ contributes nothing to $\mathcal{I}$, we have thus satisfied all requirements. On the other hand, if $N = 0$, all signs σ_j agree, and the sum $\sum_{j=1}^m \sigma_j\beta_j^{\mu_j} = \pm\sum_{j=1}^m \beta_j^{\mu_j}$ is seen to be nonvanishing on S^1. In this case, $\omega = \mathcal{I}dz/2\pi iz + \sum_{j=1}^m \lambda\beta_j^{\mu_j} + \nu_j\alpha_j$ does the job, with $sgn(\lambda) = sgn(\sigma_j)$, and $|\lambda|$ large enough so that the middle term dominates the other terms along S^1 and prevents unwanted zeros. □

Example 8. Double poles: We consider geodesics determined by differentials ω with three second order poles on the unit circle. To highlight the influence of residues and the S^1 period, the above differentials will be combined in a simple manner respecting three-fold rotational symmetry. Namely, for $j = 0, 1, 2$, let $\beta_j^{-2} = \rho_j^*\beta^{-2}, \alpha_j = \rho_j^*\alpha$, where $\rho_j(z) = \rho^j z,\ \rho = e^{2\pi i/3}$, and define

$$\omega = cid\theta + \sum_{j=0}^{2}(\nu i\alpha_j + \lambda i\beta_j^{-2}) = c\frac{dz}{z} + \nu\frac{3i(z^3+1)dz}{2z(z^3-1)} - \lambda\frac{18z^2dz}{(z^3-1)^2}$$

Here, $c = \mathcal{I}/2\pi, \nu \in \mathbb{R}$ may be specified arbitrarily, and $\lambda > 0$ is subsequently chosen large enough so that ω does not vanish on S^1 (no multiple of $\sin N\theta d\theta$ needs to be added on in this case, since ω has no odd-order poles on S^1). The resulting geodesic moves *outward* from S^1 in positive time and *inward* in negative time, while maintaining tangency to S^1 at the three cube roots of unity.

Four cases of this construction are shown in Figure 8, each with $\lambda = 1$, the first three with $\nu = 0$. In the upper left, we have also set $c = 0$, while the upper right and lower left show, respectively, $c > 0$ ($c \approx .93$) and $c < 0$ ($c \approx -.97$). Finally, dihedral symmetry is broken in the lower right with non-zero residues νi ($\nu \approx 1.75$) and $c = 0$. (We recall that ν controls the rate of change of

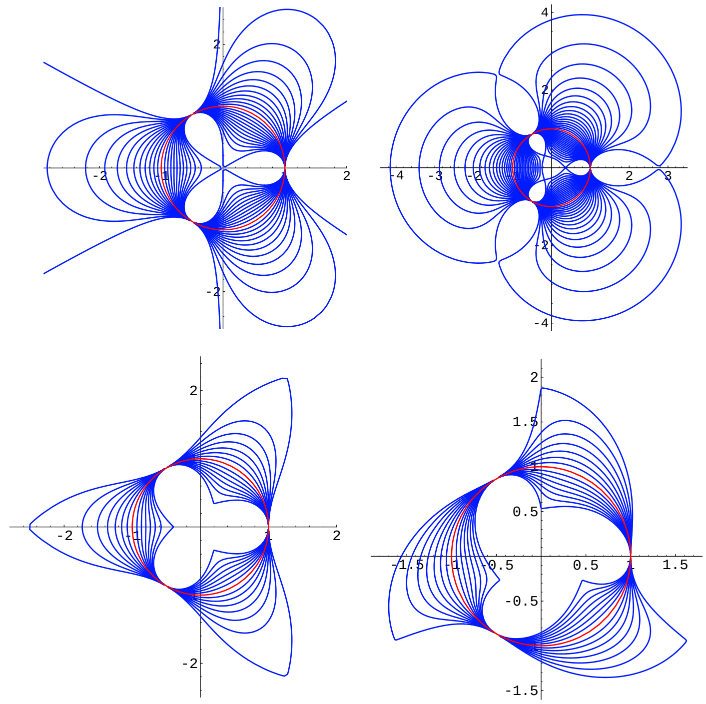

FIGURE 8. Influence of the S^1 period and residues.

the *Kasner invariant* associated with the *horn angle* formed by Γ_t and $\Gamma_0 = S^1$ at the points of tangency ρ^j—see [C-L]—but this influence is visually subtle.) The same time step has been used in each case, but the total time it takes to run into singularities increases with c and $|\nu|$. This accounts for the wide variation in number of curves displayed (and scales used).

8. Analytic diffeomorphisms of S^1

Reflection of the extended complex plane $\hat{\mathbb{C}}$ in S^1, $z \mapsto \breve{z} = 1/\bar{z} = Rz$, restricts to an involution on the group $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$ with fixed point set $S^1 = \{z \in \mathbb{C}^\times : \breve{z} = z\}$. We consider a symmetric, connected domain $U = \breve{U} \subset \hat{\mathbb{C}}$, and the group $\mathcal{M}(U)$ of meromorphic functions $g : U \to \hat{\mathbb{C}}$, g not identically zero. R induces reflection of functions in S^1 by $g \mapsto \breve{g} = \mathcal{R}(g) = R \circ g \circ R$, and $\mathcal{R} : \mathcal{M}(U) \to \mathcal{M}(U)$ is an involutive automorphism with respect to the group operation of pointwise multiplication. In this sense, $\mathcal{R}$ behaves like reflection of functions in $\hat{\mathbb{R}}$, $g(z) \mapsto \bar{g}(z) = \overline{g(\bar{z})}$. (The latter is computationally simpler in that it also satisfies $\overline{g+h} = \bar{g} + \bar{h}$ and consequently is given by conjugation of coefficients: $\sum a_n(z-z_0)^n \mapsto \sum \bar{a}_n(z-\bar{z}_0)^n$.)

$\mathcal{R}$ fixes $g \in \mathcal{M}(U)$ if and only if g belongs to the subgroup of circle-preserving functions:

$$\mathcal{M}_{\mathcal{R}}(U) = \{g \in \mathcal{M}(U) : \breve{g} = g\} = \{g \in \mathcal{M}(U) : g(S^1) \subset S^1\}.$$

Any $g \in \mathcal{M}(U)$ yields an element $\varphi \in \mathcal{M}_{\mathcal{R}}(U)$ by the formula

$$\varphi(z) = zg(z)\breve{g}(z) = zg(z)/\overline{g(1/\bar{z})}, \tag{30}$$

and $g \mapsto \varphi = \Phi(g)$ defines a mapping of $\mathcal{M}(U)$ into (*onto*, as it turns out) the *odd-degree part* of $\mathcal{M}_{\mathcal{R}}(U)$; here, degree is defined by regarding φ as a self-map of the circle. As we are presently interested in diffeomorphisms $\varphi \in H$, it will suffice to generate maps of degree one by Equation 30 using $g \in \mathcal{M}(U)$ with zero *winding number*.

Formula (30) provides a convenient way to discuss the *interpolation problem for* H, as we now demonstrate. An *N-point configuration* will refer to an N-tuple $\sigma = (\sigma_1, \ldots, \sigma_N)$ of N distinct points in $S^1 \subset \mathbb{C}$ in counterclockwise order on S^1. The space $\Sigma_N = \{\sigma\} \subset T^N$ of such configurations will be topologized as a subset of the N-torus. Let $\mathrm{Diff}_+^\omega(S^1)$ act componentwise on Σ_N, i.e., $\varphi \cdot \sigma = (\varphi(\sigma_1), \ldots, \varphi(\sigma_N))$.

Given two such configurations $\sigma = (\sigma_1, \ldots, \sigma_N)$, $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_N)$, we first show how to construct a rational function $\varphi \in \mathcal{M}_{\mathcal{R}}(\hat{\mathbb{C}})$ satisfying $\varphi \cdot \sigma = \tilde{\sigma}$. For $1 \le j \le N$, let $P_j(z)$ be the unique polynomial of degree $(N-1)$ satisfying $P_j(\sigma_k) = \delta_{jk}$ for $1 \le k \le N$. Define $p \in \mathcal{M}(\hat{\mathbb{C}})$ by

$$p(z) = \sum_{j=1}^{N} \lambda_j P_j(z), \qquad \lambda_j = \pm\sqrt{\tilde{\sigma}_j \sigma_j^{-1}} \tag{31}$$

(for the moment, either sign for λ_j will do). Now consider $\varphi = \Phi(p) \in \mathcal{M}_{\mathcal{R}}(\hat{\mathbb{C}})$. Since $p(\sigma_k) = \lambda_k$ and $\breve{p}(\sigma_k) = \breve{p}(\breve{\sigma}_k) = \breve{\lambda}_k = \lambda_k$, we have $\varphi(\sigma_k) = \sigma_k p(\sigma_k)\breve{p}(\sigma_k) = \sigma_k \lambda_k^2 = \tilde{\sigma}_k$, as required.

Proposition 5. *Let* $\mathrm{Diff}_+^\omega(S^1)$ *be the group of analytic diffeomorphisms* $\varphi(z)$ *of* S^1*. For each* N*,* $\mathrm{Diff}_+^\omega(S^1)$ *acts transitively on the space of* N*-point configurations* Σ_N*.*

Proof. By the standard argument, it suffices to show that $\mathrm{Diff}_+^\omega(S^1)$-orbits are open in Σ_N; for the connected space Σ_N is a disjoint union of its $\mathrm{Diff}_+^\omega(S^1)$-orbits, of which there can therefore be only one.

Thus, we fix a configuration $\sigma \in \Sigma_N$, and consider nearby $\tilde{\sigma} \in V \subset \Sigma_N$, where the small neighborhood V of σ will be chosen according to the following considerations. As long as $\tilde{\sigma}_k$ never equals $-\sigma_k$, for any k, all signs in Equation 31 may be defined using the principal branch of the square root $\lambda_k = \sqrt{\tilde{\sigma}_k/\sigma_k}$, an element of the right half-plane, so as to vary continuously with $\tilde{\sigma} \in V$. By this choice, the special case $\sigma = \tilde{\sigma}$ yields $\lambda_k = 1$, for all k, hence, $p(z) = \sum_{j=1}^N P_j(z) \equiv 1$ and $\Phi(p) = Id$. Further, as $\tilde{\sigma} \in V$ varies, the resulting function $p(z) = \sum_{j=1}^N \lambda_j P_j(z)$ varies continuously in the $C^1(S^1)$-topology. In fact, the assignment $\tilde{\sigma} \mapsto \mu(\tilde{\sigma}) = \varphi = \Phi(p) \in C^1(S^1)$ is continuous, so $\mu(V)$ may be assumed (by taking V small enough) to lie in $\mathrm{Diff}_+^\omega(S^1)$—indeed, within any given $C^1(S^1)$-neighborhood of the identity $\mu(\sigma) = Id \in \mathrm{Diff}_+^\omega(S^1)$. Since $\tilde{\sigma} = \mu(\tilde{\sigma}) \cdot \sigma$, it follows that the orbit $\mathrm{Diff}_+^\omega(S^1) \cdot \sigma$ contains the neighborhood $V = \mu(V) \cdot \sigma$ of σ. Since $\sigma \in \Sigma_N$ was arbitrary, we conclude that $\mathrm{Diff}_+^\omega(S^1)$-orbits are open. ☐

9. Real meromorphic differentials on S^1

A *meromorphic differential on* S^1 will refer either to a meromorphic differential $\omega = f(z)dz$ defined on $A_\epsilon = \{z \in \mathbb{C} : 1 - \epsilon < |z| < 1 + \epsilon\}$ for some $\epsilon > 0$, or to the restriction of such an ω to S^1. It will be convenient to confuse these two slightly different objects—taking full advantage of the fact that each determines the other. Indeed, we will abuse notation by writing

$$\omega = \mathbf{w}(z)dz/iz = w(\theta)d\theta \tag{32}$$

The two expressions for ω are related by formal change of variables $z = e^{i\theta}, dz = ie^{i\theta}d\theta, w(\theta) = \mathbf{w}(e^{i\theta})$, with $\mathbf{w}(z)$ the analytic continuation of $\mathbf{w}(e^{i\theta})$ to A_ϵ. One may prefer to write $z = re^{i\theta}, dz = ire^{i\theta}d\theta + e^{i\theta}dr$ and subsequently set $r = 1$ and $dr = 0$ when restricting to tangent vectors $v\partial_\theta$ along S^1. In rectangular coordinates, one drops the imaginary part of $\dfrac{dz}{iz} = \dfrac{dx + idy}{i(x+iy)} = \dfrac{xdy - ydx}{x^2+y^2} + i\dfrac{xdx + ydx}{x^2+y^2}$ when restricting to tangent vectors $\mathbf{v}(x\partial_y - y\partial_x)$. The original *analytic differential* $\mathbf{w}\frac{dz}{iz} = \alpha + i\star\alpha$ may be recovered from the *harmonic differential* $\alpha = \mathbf{w}\dfrac{xdy - ydx}{x^2+y^2}$ via the operation of *star conjugation*, $\star(adx + bdy) = -bdx + ady$ (see [Springer]).

The *singularities* of ω are the zeros and poles of $\omega = \mathbf{w}dz/iz$ on S^1. It will be assumed, henceforth that ω is not identically zero, so singularities are isolated. The *order* of ω at a point $p = e^{i\theta}$ is the integer $\mu = ord_p\omega$ in the local representation $\omega = (z-p)^\mu g(z)dz$, with $g(z)$ analytic and non-vanishing at p. For fixed ω, the value $ord_p\omega$ of the function $ord\ \omega : S^1 \to \mathbb{Z}$ is positive, negative, or zero, respectively, depending on whether p is a zero, pole, or regular point of ω. The *residue* $\nu \in \mathbb{C}$ of ω at $p = e^{i\theta}$ is well-defined by $\nu = res_p\omega = \frac{1}{2\pi i}\int_{c_\epsilon(p)}\omega$, where c is a positively oriented circle of sufficiently small radius ϵ centered at p; $\nu = a_{-1}$ in the local Laurent series representation $\omega = \sum_{n=N}^{\infty} a_n(z-p)^n dz$. In particular, $\nu = res_p\omega$ vanishes except at the finitely many poles $p_{j_1}, \dots, p_{j_n} \in S^1$.

We associate to such ω also the S^1*-period* $\int_{\gamma_\epsilon}\omega$ obtained by integrating around a slightly smaller circle, $\gamma_\epsilon = (1-\frac{\epsilon}{2})e^{i\theta}$. Finally, in case ω has zeros $p_{k_1}, \dots, p_{k_m} \in S^1$, assumed to be indexed in counterclockwise cyclic order, we define *subperiods* $\int_{\gamma_s}\omega$ by integrating along a suitable path γ_s joining p_{k_s} to $p_{k_{s+1}}$; γ_k starts at p_{k_s}, goes radially inward a distance $\epsilon/2$, then counterclockwise along a portion of γ_ϵ, then radially out to $p_{k_{s+1}}$. When $m > 0$, the sum of subperiods equals the S^1-period.

Of particular interest will be the *real* meromorphic differentials $\omega = w(\theta)d\theta$ on S^1; here $w(\theta) \in \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ for all θ, and $w(\theta)$ may be regarded as a real analytic function on S^1 (helping to motivate the above notation). In this case, the residues $\nu_1, \dots, \nu_n$ are automatically real; further, these values determine the imaginary parts $Im[\int_{\gamma_\epsilon}\omega]$, $Im[\int_{\gamma_s}\omega]$ of the above integrals. These claims follow (via *Cayley transformation*, $w = \dfrac{z-i}{z+i}$, $z = i\dfrac{w+1}{1-w}$) from corresponding facts for real meromorphic differentials on the extended real line $\hat{\mathbb{R}}$.

Alternatively, the antiholomorphic involution in the Riemann sphere $\hat{\mathbb{C}}$ given by reflection in the unit circle $z \mapsto \check{z} = 1/\bar{z}$ may be used—just as well as complex conjugation—to define a *real structure* on $\hat{\mathbb{C}}$ and associated spaces. Thus, one may *conjugate* functions $f(z)$, and differentials $\omega = f(z)dz$ according to the formulas:

$$f^*(z) = 1/\check{f}(z) = \overline{f(1/\bar{z})}, \quad \omega^* = f^*(z)dz^* = -f^*(z)dz/z^2 \tag{33}$$

Unlike *star conjugation* (which satisfies $\star\star\omega = -\omega$), $\omega \mapsto \omega^*$ is an involution and behaves formally like conjugation w.r.t. the real line: $\bar{\omega} = \bar{f}(z)dz = \overline{f(\bar{z})}dz$. In particular, if $\gamma^* = \check{\gamma}$ is the reflection of an oriented curve γ in S^1, then one directly verifies that $\int_{\gamma^*}\omega^* = \overline{\int_\gamma \omega}$.

In the case of a real differential $\omega = w(\theta)d\theta = \mathbf{w}(z)dz/iz$, we have $\omega^* = \omega$. For curves satisfying $\gamma^* = -\gamma$, e.g., for a small circle meeting S^1 orthogonally, it follows that $\int_\gamma \omega = -\int_{\gamma^*}\omega^* = -\overline{\int_\gamma \omega}$ is imaginary; thus, any residues of ω on S^1 must be real, as claimed above. Similarly, by taking a contour γ which mostly follows S^1, but makes small inward detours around the poles p_{j_r}, one obtains $2iIm[\int_{\gamma_\epsilon}\omega] = 2iIm[\int_\gamma \omega] = \int_\gamma \omega - \int_{\gamma^*}\omega^* = -2\pi i \sum Res[\omega; p_{j_r}]$. The quantities $Im[\int_{\gamma_s}\omega]$ are handled similarly. In view of these facts, the S^1*-period* $\mathcal{I}$ and *subperiods* $\mathcal{I}_s$ of a real meromorphic differential on S^1 will refer, henceforth, to the respective real parts $\mathcal{I} = Re[\int_{\gamma_\epsilon}\omega]$ and $\mathcal{I}_s = Re[\int_{\gamma_s}\omega]$.

For such ω with no poles on S^1, we have $\mathcal{I} = \int_{S^1}\omega$. More generally, if ω has at most simple poles on S^1 then $\mathcal{I}$ may be computed as $\mathcal{I} = PV\{\int_{S^1}\omega\}$—the *Cauchy principal value integral* defined by passing to the limit $\epsilon \to 0$ in the integral of ω along S^1, omitting arcs $\theta_j - \epsilon < \theta < \theta_j + \epsilon$ about each pole $p_j = e^{i\theta_j}$. Such integrals are discussed, e.g., in [Ablowitz-Fokas], where one may also find the following elementary result: If $f(z)dz$ has a simple pole at z_0 and $C(\epsilon)$ denotes a circular arc centered at z_0 with angular measure ϕ and (variable) radius ϵ, then $\lim_{\epsilon\to 0}\int_{C(\epsilon)} f(z)dz = i\phi Res[f(z)dz, z_0]$. This may be applied in the present context by deforming γ_ϵ to a new contour $\tilde{\gamma}_\epsilon$ which coincides with S^1 except for "inward" detours $C_j(\epsilon)$ around the poles, $C_j(\epsilon)$ being nearly half of a circle of radius ϵ, centered at p_j. In the limit $\epsilon \to 0$, the detours contribute the imaginary part of $\int_{\tilde{\gamma}_\epsilon}\omega$ (the residues of ω being real), while the real part is given by the above Cauchy principal value integral over S^1. Thus, we have heuristically explained the term S^1-*period* (though the Cauchy principal value interpretation of $\mathcal{I}$ is no longer valid when ω has higher order poles).

We need to consider one additional type of data for such $\omega = w(\theta)d\theta$, the *polarity* (or *sign*) of the j^{th} singularity $p_j = e^{i\theta_j}$: for θ near θ_j, the sign is given by $\sigma_j = \pm 1 = sgn\ w(\theta)(\theta - \theta_j)^{\mu_j}$. In particular, the polarity of a simple pole agrees with the sign of the residue, as one may easily verify. Further, the polarity of any singularity is invariant under orientation-preserving analytic diffeomorphisms of S^1.

Now observe that the sign of the real function $w(\theta)$ changes precisely at odd-order singularities. Consequently, the sequence $\sigma_1, \sigma_2, \ldots, \sigma_m$ jumps up or down precisely at odd-order singularities as all singularities are visited in counterclockwise order. If there is at least one simple pole, it follows that the sequence $\{\sigma_j\}$ is uniquely determined by the values $\{\mu_j\}$, $\{\nu_j\}$; otherwise, the sequence is determined only up to an over-all factor ± 1. The data $\{\mu_j, \nu_j, \sigma_j\}$ for a real, meromorphic differential $\omega = w(\theta)d\theta$ on S^1 is subject to the two *polarity constraints* (PC):

$$\sigma_j = (-1)^{\mu_j}\sigma_{j-1}; \qquad \mu_j = -1 \Rightarrow\ \sigma_j = sgn\ \nu_j \tag{34}$$

Note that (PC) imposes constraints even on the abridged data $\{\mu_j, \nu_j\}$; e.g., the sum $\sum_{j=1}^m \mu_j$ must be even, and the residues ν_j, ν_{j+1} at two adjacent simple poles have opposite sign, etc.

10. Equivalence of real meromorphic differentials on S^1

As shown in [Moser], two smooth volume forms ω and $\tilde\omega$ on a compact, connected manifold M are *equivalent*—they are related by pull-back, $\omega = h^*\tilde\omega$, under an orientation-preserving diffeomorphism h of M—if and only if ω and $\tilde\omega$ have the same volume $\int_M \omega = \int_M \tilde\omega$. (The necessary condition follows at once from the change of variables formula.) The second of the two main arguments used in [Moser] (the "Hodge-theoretic" one) may be adapted to the present context of real meromorphic differentials on S^1, as we now demonstrate.

Lemma 1. *Let $\omega_r = w_r d\theta$, $r = 0, 1$, be two real meromorphic differentials on S^1 satisfying:*

(1) *The ratio $q(z) = \omega_1/\omega_0$ defines a positive analytic function on S^1; i.e., ω_0, ω_1 have the same singularities $p_1, \ldots, p_m \in S^1$, with equal orders μ_j and polarities σ_j.*
(2) *ω_0, ω_1 have equal residues ν_j, S^1-periods $\mathcal{I} = Re[\int_{\gamma_\epsilon} \omega_r]$, and sub-periods $\mathcal{I}_k = Re[\int_{\gamma_k} \omega_r]$ (in the event there are zeros).*

Then ω_0 and ω_1 are analytically equivalent—in fact, diffeotopic: the linear isotopy of differentials, $\omega_t = t\omega_1 + (1-t)\omega_0$, is realized by pull-back, $\omega_0 = h_t^*\omega_t$, via a diffeotopy $h_t : [0,1] \times S^1 \to S^1$.

Proof. Each intermediate $\omega_t = w_t d\theta$ in the isotopy is of the same type and shares the same data as ω_0, ω_1. In particular, the quotient $q_t = \omega_t/\omega_0$ defines, for each fixed $t \in [0,1]$, a positive, real analytic function on S^1. By (2), the time-derivative $\dot\omega = \frac{d}{dt}\omega_t = \omega_1 - \omega_0$ has vanishing periods on $A_\epsilon \backslash P$, an annular neighborhood of S^1 with polar set of ω_t removed. Thus, $\dot\omega$ is exact and may be expressed as $\dot\omega = d\alpha$ for some real meromorphic function $\alpha : S^1 \to \mathbb{R} \cup \{\infty\}$. In case there are zeros, we choose α to vanish at one of them; hence, by (2), it vanishes on all zeros of ω_t.

Differentiating the desired equation $\omega_0 = h_t^*\omega_t$ and suppressing the $t-$subscripts, we obtain:

$$0 = \frac{d}{dt}h^*\omega = \frac{d}{dt}h' w \circ h d\theta = [\dot h' w \circ h + h'\dot w \circ h + h' w' \circ h \dot h]d\theta$$

Using $[\alpha \circ h]' = \alpha' \circ h h' = h'\dot w \circ h$ to replace the middle term, the above equation may be expressed more compactly as: $0 = [\dot h w \circ h + \alpha \circ h]'$. In fact, we aim to make the bracketed quantity itself vanish, that is, to solve the following ODE:

$$\frac{dh}{dt} = -\alpha(h)/w(h), \qquad h_0 = Id \tag{35}$$

Assuming Equation 35 has been solved for analytic diffeomorphisms h_t, $0 \leq t \leq 1$, we then have $h_t^*\omega_t = const$ and $\omega_0 = h_0^*\omega_0 = h_1^*\omega_1$, as required.

To show that the above ODE is suitably well-behaved (and to see why all hypotheses are needed), consider the meromorphic differentials $\eta = -\omega/\alpha$, $0 \leq t \leq 1$, at singular and non-singular points of α. If $z \in S^1$ is not a zero or pole of α, then $z \notin Z$, so $\eta = -\omega/\alpha \neq 0$ at z. On the other hand, if $z \in S^1$ is a zero or pole of α, then $d\alpha/\alpha$ has a (simple) pole at z. Since the quotient $\omega/d\alpha = \omega/(\omega_1 - \omega_0) = q_t/(q_0 - 1)$ is nonvanishing on S^1 for $0 \leq t \leq 1$, it follows that $\eta = -\dfrac{\omega}{d\alpha}\dfrac{d\alpha}{\alpha}$ has a pole at z. We conclude that η is nonvanishing on S^1 for $0 \leq t \leq 1$ (and one can show that η has poles at all singularities of ω_t).

Dualizing $\eta = -wd\theta/\alpha$, we thus obtain the (time-dependent) real analytic vectorfield on S^1, $X = -\dfrac{\alpha}{w}\partial_\theta$, defining Equation 35. By the theory of ODE's, X generates an analytic diffeotopy h on the compact manifold S^1 (see, e.g., [Hirsch] for the smooth theory, and [Cartan], [Dieudonne] for analytic dependence on initial conditions). ☐

Theorem 2. *i) Two real, meromorphic differentials on S^1 are equivalent if and only if they have identical order-residue-polarity-period data, $\mu_j, \nu_j, \sigma_j, \mathcal{I}_k, \mathcal{I}$, for suitable counterclockwise orderings of singularities.*

ii) For such differentials with no zeros on S^1, all equivalence classes are represented by restriction to S^1 of rational differentials on $\hat{\mathbb{C}}$, as the latter achieve arbitrary data $\mu_j, \nu_j, \sigma_j, \mathcal{I}$, subject only to the polarity constraints (PC) and, when there are no poles, $\mathcal{I} \neq 0$.

iii) The Adjoint orbits $Ad_H A$ are characterized by data of the dual meromorphic differentials as in ii); in particular, each is represented by restriction to S^1 of a rational vectorfield on $\hat{\mathbb{C}}$.

Proof. We combine results discussed above. For the nontrivial direction of i), suppose two differentials share data $\mu_j, \nu_j, \sigma_j, \mathcal{I}_k, \mathcal{I}$, for some counterclockwise orderings of singularities. Then by Proposition 5 we can pull back one of the differentials by some $h \in H$ to bring locations of corresponding singularities into agreement. Then Lemma 1 implies the two differentials are equivalent (in fact they are *diffeotopic*).

Subsequently, ii) follows the constructions of §7, which show how all data $\mu_j, \nu_j, \sigma_j, \mathcal{I}$ consistent with (PC) may be achieved by rational differentials which are real and nonvanishing along S^1. More complicated constructions appear to be needed for explicit representation of dfferentials in the general case (which we do not need).

Finally, ii) yields iii) via duality $A = a(\theta)\partial_\theta \;\leftrightarrow\; \omega = d\theta/a$; this bijection between $\mathfrak{h}$ and nonvanishing, real meromorphic differentials along S^1, interwines λ_{h*} and pull-back by h^{-1}. Explicitly, if $A = a(\theta)\partial_\theta$ and $\tilde{A} = Ad_h A = \tilde{a}(\theta)\partial_\theta$, then Equation 20 gives $\tilde{a}(h(\theta)) = h'(\theta)a(\theta)$; hence, $\omega = d\theta/a(\theta) = h'(\theta)d\theta/\tilde{a}(h(\theta)) = h^*(d\theta/\tilde{a}(\theta)) = h^*\tilde{\omega}$. ☐

Remark 6. Simple heuristics further illuminate the nature of adjoint orbits of H. By Equation 22, $0 \neq A \in \mathfrak{h}$ spans the kernel of $B \mapsto ad_B A$. The one parameter group $\{e^{tA}\}$ is the continuous part of the isotropy subgroup $H_A = \{B \in H : Ad_B A = A\}$. For A nonvanishing on S^1, we have $H_A \simeq S^1$; the corresponding orbit $Ad_H A$ may be identified with the homogeneous space H/S^1, already mentioned in §5. On the other hand, if A vanishes on S^1, its orbit looks like $H/\mathbb{R}$ or, in exceptional cases, $H/(\mathbb{R} \times \mathbb{Z}_n)$, where $\mathbb{Z}_n$ is a finite cyclic group which permutes zeros of A.

Not surprisingly, coadjoint orbits of H are more often discussed (mostly in the context of the *Virasoro algebra* $\hat{\mathfrak{h}}$ defined by central extension of $\mathfrak{h}$—see, e.g, [Witten]). Elements of the *regular dual space* $\mathfrak{h}^*_{reg}$ of $\mathfrak{h}$ may be represented by quadratic differentials $p = p(\theta)d\theta^2$ via the pairing $(a\partial_\theta, p) = p(A) = \int_{S^1} apd\theta$. When $p = w^2 d\theta^2$ is nonvanishing along S^1, the kernel of $B \mapsto ad^*_B p = -(2pb' + p'b)d\theta^2$ is spanned by $\frac{1}{w}\partial_\theta \in \mathfrak{h}$, and one has $Ad^*_H p \simeq H/S^1$; thus, certain adjoint and coadjoint orbits resemble each other. (In the context of $\hat{\mathfrak{h}}$, the symplectic nature of H/S^1 leads to a well-known, remarkable derivation of the *Korteweg-de Vries equation.* Elsewhere, we will explore implications of such structure on the space of symmetric, singular ring domains.)

For nonvanishing $A \in \mathfrak{h}$, we may define $\mathbf{p}_A \in \mathfrak{h}^*_{reg}$ by the formula $\mathbf{p}_A(c) = \int_0^{2\pi} c/a^2 d\theta$, as the last paragraph suggests; but in general, the poles of $d\theta/a^2$ resulting from the zeros of A interfere with the interpretation of this formula. Similarly, given $p \in \mathfrak{h}^*_{reg}$ with a zero on S^1, p is not stabilized by any continuous vectorfield and $Ad^*_H p$ looks like (a finite quotient of) H itself—unlike *any* adjoint orbit! The difference between adjoint and coadjoint actions has already been reflected, above, in the lack of a pseudo-Riemannian structure on Λ.

Remark 7. The kernel of $ad_A : \mathfrak{g} \to \mathfrak{g}$, has a simple interpretation as Lie algebra of the S^1-*local isometry group* of the foliated flat geometry defined by $p = \omega^2 = dz^2/(f(z))^2$, where $A = f\partial_z$. (Such isometries are only required to be defined on a punctured neighborhood $A_r \setminus \{z_1, \ldots, z_n\}$ of S^1.) Namely, for any constant $\lambda \in \mathbb{C}^\times$, the vectorfield $Re(\lambda W)$ generates a one-parameter local group $\{h_t\}$ satisfying $Ad_{h_t} W = W$, hence, $h_t^* \omega = \omega$, so $g = \omega\bar{\omega}$ and the horizontal foliation remain fixed.

If one relaxes the requirement that an isometry should preserve trajectories of $p = \omega^2$—that is, if one regards $(A_r \setminus \{z_1, \ldots, z_n\}, g = \omega\bar{\omega})$ simply as a Riemannian manifold—then there are exceptional examples with *three*-dimensional local isometry groups. To find such examples, first suppose a conformal map h preserves the metric: $h^* g = g$, i.e., $\tilde{p} = h^* p = e^{2i\theta} p$. Then the real function θ must be constant, as it is locally the ratio of holomorphic functions $\tilde{P}(z), P(z)$, and $\tilde{p}$ therefore belongs to the *conjugate family* $\{e^{2i\theta} p\}$, $0 \le \theta < 2\pi$. If a holomorphic vectorfield $B = b\partial_z$ generates a one-parameter group h_t of such maps, it induces a variation of the form $e^{i\theta(t)}\omega$. In this case B satisfies $ad_B W = icW$, where $W = f\partial_z$ is the holomorphic vectorfield dual to $\omega = dz/f(z)$ and c is a real constant.

In general, the formula $b = a\int c/a^2$ defines a solution $B = b\partial_\theta \in \mathfrak{h}$ to the equation $ad_B A = C$ (uniquely up to multiples of A)—provided the latter antiderivative yields a single-valued holomorphic function b along S^1. (In the case of nonvanishing A, it follows that $Image(ad_A) = Kernel(\mathbf{p}_A)$, as $\mathbf{p}_A(c) = \int_0^{2\pi} c/a^2 d\theta = 0$ is the condition for periodicity, $b(0) = b(2\pi)$.) For the special equation $ad_B W = icW$, the formal solution $b(z) = icf(z)\displaystyle\int_{z_0}^{z} \frac{d\zeta}{f(\zeta)}$ yields a well-defined holomorphic vectorfield $B = b\partial_z$ near S^1 in the exceptional case where $\omega = dz/f(z)$ has vanishing S^1-period $\mathcal{I} = Re[\int_{\gamma_\epsilon} \omega]$, and vanishing residues at all the poles $z_1, \ldots, z_n \in S^1$. In this case, B is unique up to addition of terms λW already considered, and the combinations $\{cB + \lambda W\}$ generate a *three*-dimensional local isometry group. The simplest example of such exceptional behavior was already given in the last part of Example 1. More complicated examples may easily be constructed as indicated, and it is amusing to picture Euclidean isometries associated with the various "infinities" z_j coordinating to give an isometry on a neighborhood $A_r \setminus \{z_1, \ldots, z_n\}$.

References

[Ablowitz-Fokas] M. Ablowitz and A. Fokas, **Complex Variables**, *Cambridge Texts in Applied Mathematics*, Cambridge University Press (1997).

[Ahlfors] Lars Ahlfors, **Complex Analysis** (*First Edition*), *International Series in Pure and Applied Mathematics*, McGraw-Hill (1953).

[Byrd-Friedman] Paul F. Byrd and Morris D. Friedman, **Handbook of Elliptic Integrals for Engineers and Physicists**, Springer-Verlag (1954).

[C-L] Annalisa Calini and Joel Langer, Schwarz reflection geometry I: continuous iteration of reflection, **Math. Z., 244** (2003) pp. 775–804.

[Cartan] Henri Cartan, **Elementary Theory of Analytic Functions of One or Several Complex Variables**, Editions Scientifiques Herman (1963), translation, Addison-Wesley.

[Davis] Philip J. Davis, **The Schwarz Function and its Applications**, *The Carus Mathematical Monographs, No. 17*, The Mathematical Association of America (1974).

[Dieudonne] J. Dieudonné, **Foundations of Modern Analysis**, *Pure and Applied Mathematics Series, v. 10 I*, Academic Press (1969).

[Guest] Martin Guest, **Harmonic Maps, Loop Groups, and Integrable Systems**, *London Mathematical Society Student Texts, No. 38*, Cambridge University Press (1997).

[Hirsch] Morris W. Hirsch, **Differential Topology**, Graduate Texts in Mathematics **33**, Springer-Verlag, (1976).

[Kirillov] A.A. Kirillov, Geometric approach to discrete series of unirreps for Vir, ESI preprint (1995).

[Kobayashi-Wu] S. Kobayashi and H. Wu, **Complex Differential Geometry**, Birkhäuser (1983).

[Krichever-Novikov] I. M. Krichever and S. P. Novikov, Algebras of Virasoro type, Riemann surfaces and structures of the theory of solitons, Translated from **Funktsional'nyi Analiz i Ego Prilozheniya, 21**, Plenum Publishing Corporation (1987) pp. 46–63.

[Lehto-Virtanen] , O. Lehto and K. I. Virtanen, **Quasiconformal Mappings in the Plane**, Springer-Verlag (1973).

[Loos] Ottmar Loos, **Symmetric Spaces I: General Theory**, W. A. Benjamin, Inc. (1969).

[McMullen] C. McMullen, **Complex Dynamics and Renormalization**, Annals of Mathematics Studies **135**, Princeton University Press (1994).

[Milnor] John Milnor, Remarks on infinite dimensional Lie groups, in *Relativity, Groups and Topology II*, Les Houghes Session XL, 1983, edited by B. S. de Witt and R. Stora, North-Holland, Amsterdam, 1984.

[Moser] Jürgen Moser, On the volume elements on a manifold, TAMS, Vol. 120, Issue 2 (1965), pp. 286-294.

[Mucino-Raymundo] Jesus Muciño-Raymundo, Complex structures adapted to smooth vector fields, **Math. Ann. 322** (2002), pp. 229-265.

[Shapiro] Harold S. Shapiro, **The Schwarz Function and its Generalization to Higher Dimensions**, *University of Arkansas Lecture Notes in the Mathematical Sciences, 9,* John Wiley & Sons, Inc., (1992).

[Sharpe] R. W. Sharpe, **Differential Geometry**, Springer-Verlag, (1997).

[Springer] George Springer, **Introduction to Riemann surfaces**, Addison-Wesley, Inc. (1957).

[Strebel] Kurt Strebel, **Quadratic Differentials**, Springer-Verlag, (1984).

[Zygmund] A. Zygmund, **Trigonometric Series, Volume II**, Cambridge University Press, (1959).

[Witten] Edward Witten, Coadjoint orbits of the Virasoro group, **Comm. Math. Phys. 114** (1988).

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