Bäcklund transformations and knots of constant torsion

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Abstract The Bäcklund transformation for pseudospherical surfaces, which is equivalent to that of the sine-Gordon equation, can be restricted to give a transformation on space curves that preserves constant torsion. We study its effects on closed curves (in particular, elastic rods) that generate multiphase solutions for the vortex filament flow (also known as the Localized Induction Equation). In doing so, we obtain analytic constant-torsion representatives for a large number of knot types.

Introduction

Soliton equations have become a familiar presence in the differential geometry of curves and surfaces. The description of pseudo-spherical surfaces and their asymptotic lines in terms of the sine-Gordon equation dates back nearly a century. Of roughly the same date is the derivation (see [Ri]) of the Localized Induction Equation (LIE), which provides one of the richest examples of connection between curve geometry and integrability.

The understanding of this connection has progressed in recent years along different directions. On the one hand, several fundamental properties of soliton equations have been given a geometrical realization; in the case of the LIE, its bi-hamiltonian structure and recursion operator, its hierarchy of constants of motion, and its relation to the nonlinear Schrödinger equation possess a natural geometric interpretation [L-P1].

On the other hand, some well-known classes of curves in differential geometry have been identified with solutions of integrable equations: for example, elastic curves and center-lines of elastic rods are among the solitons for the LIE, curves of constant torsion correspond to characteristics for the sine-Gordon equation, and planar and spherical curves are associated with solutions of the mKdV hierarchy ([G-P],[L-P2],[D-S]).

A third direction of research ([C],[M-R]) concerns the topological properties of closed curves that arise as solutions to soliton equations. A major question is whether the presence of infinitely many symmetries and the associated sequence of integral invariants may be related to knot invariants; in fact, one hopes that the knot types of the periodic analogues of soliton solutions (the so-called multi-phase solutions) can be described using methods of integrable systems, such as the Floquet spectrum and Bäcklund transformations. We remark that the knot

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types of closed elastic curves, which turn out to generate two-phase solutions of the LIE, were classified by Langer and Singer [L-S1] with the aid of infinitesimal symmetries related to flows in the LIE hierarchy.

In this work we illustrate these different approaches through the study of a unifying example. First, we propose a geometric realization of the Bäcklund transformation for the sine-Gordon equation in the context of curves of constant torsion, by restricting the classical Bäcklund transformation for pseudo-spherical surfaces [C-T] to a constant-torsion-preserving transformation between asymptotic lines. Because it is not necessary to embed the curve in a surface to carry out the transformation, this is also a tool for obtaining new interesting curves of constant torsion.

Secondly, we are interested in curves that generate special solutions (in particular, solitons and their periodic counterparts) of integrable equations and whose analytic expressions and general properties are well-known. This choice is particularly significant for closed curves, since multi-phase solutions are dense in the space of all periodic solutions: it is therefore important to understand their geometrical and topological properties. The simplest non-planar constant torsion curves in this class are the centerlines of elastic rods. (Elastic rods, which also generate two-phase solutions of the LIE, are critical curves for the elastic energy (total squared curvature), subject to fixed total torsion and fixed length.)

Finally, we study the topological properties of curves obtained by means of (single and iterated) Bäcklund transformations of constant torsion elastic rods. The behaviour of these transformations is diverse: we show that a single Bäcklund transformation preserving closure leaves the class of constant torsion elastic rods invariant, and we compute an interesting formula relating the linking number of a curve with its transform and the self-linking of the original curve. The iterated Bäcklund transformation instead allows one to leave the class of elastic rods and to produce a variety of phenomena such as knotting, self-intersections and unknotting. Effects of Bäcklund transformations on the multi-phase solutions of the sine-Gordon equation have been discussed in the work by Ercolani, Forest and McLaughlin [E-F-M], which uses a combination of Floquet theory for the associated spectral problem and methods of algebraic geometry. We present here the first concrete realization of their conclusions in a geometric setting, and give simple direct proofs for the case of elastic rods of constant torsion.

This article is structured as follows. In order to implement the Bäcklund transformation for pseudospherical surfaces restricted to constant torsion curves, we need to solve an associated Riccati equation. To overcome this difficulty, we prove in Part I that, if we know the expression of the Frenet frame of a given curve that depends analytically on the constant torsion τ, then we can obtain the solution to the Riccati equation by analytic continuation to an imaginary value of the torsion. In other words, for any curve whose Frenet frame can be continued analytically as a function of τ, with fixed curvature function, the Bäcklund transformation is directly computable. This is illustrated for closed elastic rods of constant torsion. Using Langer and Singer's solution of the Euler-Lagrange equations [L-S4], we derive explicit formulas for these curves and their Frenet frames, and we also describe their knot types. We then show that the condition that the Bäcklund transformation give another closed curve forces the new curve to be an elastic rod that is congruent to the original. However, such a closed Bäcklund transformation carries topological information by providing a measure of the knottedness of the initial elastic rod. In §1.6 we show this by relating the linking number of the curve with its transform to the self-linking of the original curve. Part I ends with a discussion of, and a precise conjecture about, the effect of Bäcklund transformations on constant torsion n-phase (and n-soliton) solutions of the LIE hierarchy. The conjecture, which is easily verified for the first few values of n, concerns the form of certain Killing fields associated with n-soliton curves, and it implies that n-solitons are in general taken to $(n+1)$-solitons.

Part II concerns the iterated Bäcklund transformation, which consists of two successive Bäcklund transformations using related solutions of the Riccati equations. For the purpose of constructing iterations, we conveniently rederive the original Bäcklund formula in terms of a gauge transformation of the sine-Gordon linear system. (This is simply the spatial part, in characteristic coordinates, of the Lax pair, and is equivalent to our Riccati equation.) Using this formulation, a simple algebraic procedure produces the iterated formula which we apply to constant torsion elastic rods to produce a variety of interesting curves. The closure condition, developed using standard arguments of Floquet theory, is in this case less restrictive. As a consequence, we prove that, beyond producing congruent elastic rods, an iterated Bäcklund transformation can take constant torsion two-phase solutions to multi-solitons whose knot type differs dramatically from that of the original curve. All three phenomena occur: knotting, self-intersections and unknotting, all related to the Floquet spectrum of the linear system associated to the initial curve, as illustrated and discussed in the concluding section.

The investigations that we report here were initiated by, and advanced with the steady encouragement of, the geometry group at Case Western Reserve University (Joel Langer, David Singer, and the authors), with important contributions from Ron Perline of Drexel University. In particular, Perline first asked if there exist constant torsion solitons for the LIE; Singer classified the elastic rods of constant torsion; and, Langer and Perline developed the theory of planar-like solitons and their associated Killing fields which allows us to connect our Bäcklund transformations with the LIE hierarchy. We gratefully acknowledge these contributions, and the lively discussions that helped to shape this work.

I. The single Bäcklund transformation

1.1 Bäcklund transformations and the Frenet frame

In classical differential geometry, a Bäcklund transformation takes a given pseudospherical (i.e. constant negative Gauss curvature) surface to a new pseudospherical surface. As explained by Chern and Terng [C-T], the new surface is connected to the old surface by line segments that are tangent to both surfaces, of a fixed length, and such that the angle between the surface normals at corresponding points is also constant. Moreover, the Bäcklund transformation takes asymptotic lines to asymptotic lines. Since the asymptotic lines on a pseudospherical surface have constant torsion, it is not surprising that we can restrict the Bäcklund transformation to get a transformation that carries constant torsion curves to constant torsion curves.

Theorem 1.1. *Let $\gamma(s)$ be a smooth curve of constant torsion τ in $\mathbb{R}^3$, parametrized by arclength s. Let T, N, B be a Frenet frame, and $\kappa(s)$ the curvature of γ. For any constant C, let $\beta = \beta(s; \kappa(s), C)$ be a solution of the differential equation*

$$\frac{d\beta}{ds} = C \sin\beta - \kappa. \tag{1}$$

Then the curve

$$\widetilde{\gamma}(s) = \gamma(s) + \frac{2C}{C^2 + \tau^2}(\cos\beta T + \sin\beta N) \tag{2}$$

is a curve of constant torsion τ, also parametrized by arclength.

Proof. Since the angle between the normals of the two pseudospherical surfaces is constant at corresponding points, it follows that the angle between the binormals of γ and $\widetilde{\gamma}$ is constant. In fact, one computes the Frenet frame along $\widetilde{\gamma}$:

$$\begin{aligned}
\widetilde{T} &= T + (1 - \cos\theta)\sin\beta(\cos\beta N - \sin\beta T) + \sin\theta \sin\beta B \\
\widetilde{N} &= N - (1 - \cos\theta)\cos\beta(\cos\beta N - \sin\beta T) - \sin\theta\cos\beta B \\
\widetilde{B} &= \cos\theta B + \sin\theta(\cos\beta N - \sin\beta T),
\end{aligned}$$

where $\tan(\theta/2) = C/\tau$. Then the torsion is τ and the curvature is

$$\widetilde{\kappa} = \kappa - 2C\sin\beta. \tag{3}$$

□

We should point out that we will assume our curves have a smooth *generalized Frenet frame*, that is, smooth orthonormal vector fields T, N, B that satisfy the Frenet equations with respect to smooth functions κ, τ. In particular, κ is allowed to change sign. Given an orientation of γ, T is well defined, and N, B, κ are well-defined along such a curve up to multiplying all three by -1. If the curve is closed of length L, then our assumptions imply that there must either be a smooth Frenet framing by vector fields of period L, or a framing by vector fields of period $2L$ such that N, B, κ change by a minus sign after one circuit of the curve. (We will refer to these closed curves as 'even' and 'odd', respectively.)

Example. To compute the Bäcklund transformation of a constant torsion curve, one needs to solve the differential equation (1), which is equivalent to a Riccati equation when one changes variables to $y = \tan(\beta/2)$. When κ is equal to a constant κ_0, the curve γ is a helix, and the ODE is easy to solve. For $C < \kappa_0$, the solutions are periodic, and one can arrange that the period is commensurable with the translational period of the helix, yielding a curve $\widetilde{\gamma}$ that is periodic up to translation (see Figure 1.1(b)). For $C \geq \kappa_0$, the solutions are asymptotically constant, and yield curves $\widetilde{\gamma}$ that are asymptotic to a helix (see Figure 1.1(c)). For $C = \kappa_0$, the curvature of $\widetilde{\gamma}$ is a rational function of arclength:

$$\widetilde{\kappa}(s) = \kappa_0 \frac{1 - x^2}{1 + x^2}, \qquad x = \kappa_0 s.$$

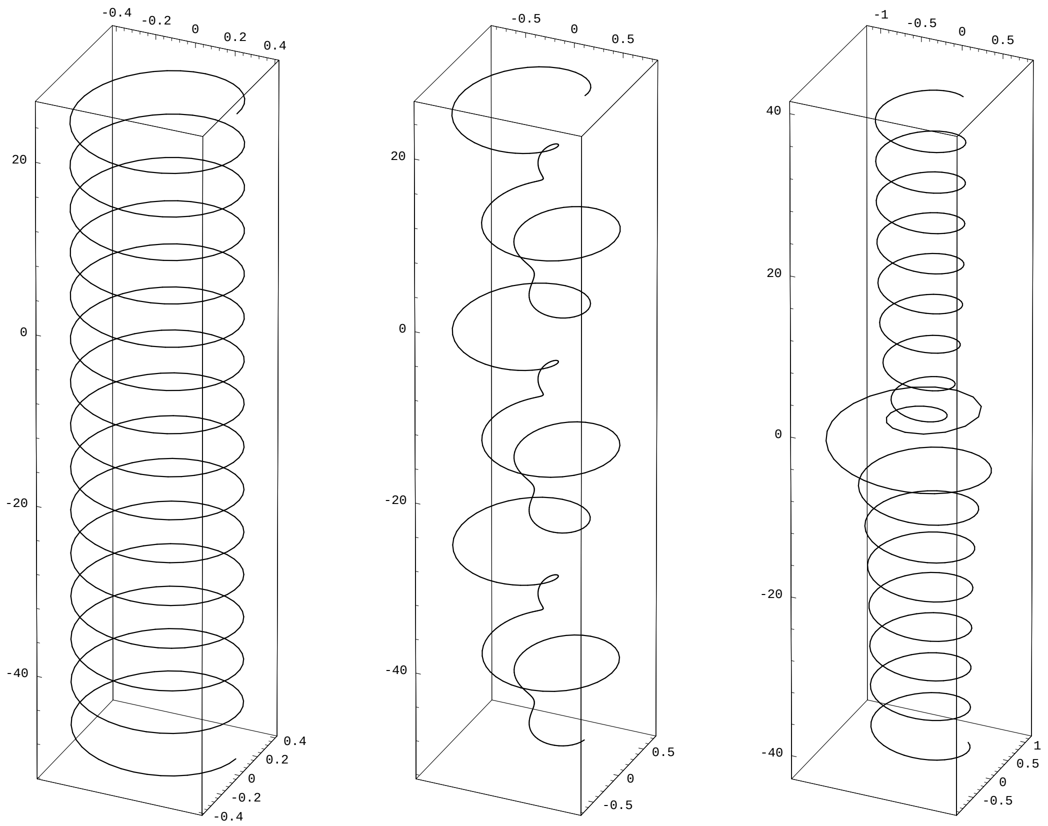

FIGURE 1.1. A helix and its periodic and asymptotically helical Bäcklund transformations

The Bäcklund transformation for pseudospherical surfaces is, of course, equivalent to the Bäcklund transformation for the sine-Gordon equation. If s and t are arclength coordinates along the asymptotic lines, and θ is the angle between the asymptotic lines, then the Codazzi equations imply $\theta_{xt} = \sin\theta$. In terms of $\varphi = \theta/2$, the Bäcklund transformation for sine-Gordon is

$$(\widetilde{\varphi} - \varphi)_s = C \sin(\widetilde{\varphi} + \varphi), \qquad (\widetilde{\varphi} + \varphi)_t = C^{-1} \sin(\widetilde{\varphi} - \varphi).$$

If T, N are chosen along an s-curve so as to agree with the orientation $\partial/\partial t \wedge \partial/\partial s$ on the surface, then $\theta_s = \kappa(s)$. Once we make this identification, then (1) follows if $\beta = -(\varphi + \widetilde{\varphi})$.

Along with applying the Bäcklund transformation to constant torsion curves, we can also apply the nonlinear superposition principle for solutions of the sine-Gordon equation [Ro]. (For surfaces, this is Bianchi's theorem of permutability [E].) Suppose $\beta_1 = \beta(s; \kappa(s), C_1)$ and $\beta_2 = \beta(s; \kappa(s), C_2)$ generate two Bäcklund transformations, with corresponding curvature functions $\kappa_1(s), \kappa_2(s)$ given by (3). Then particular solutions $\beta_{12} = \beta(s; \kappa_1(s), C_2)$ and $\beta_{21} = \beta(s; \kappa_2(s), C_1)$ to successive Bäcklund transformations are given by

$$\beta_{12} - \beta_1 = \beta_{21} - \beta_2 = 2 \arctan \left[\frac{C_1 + C_2}{C_1 - C_2} \tan \left(\frac{\beta_1 - \beta_2}{2} \right) \right]. \tag{4}$$

Now suppose that we can solve (1) for some initial curvature function and all values of C, as we can for the helix. Then in principle we can use (4) to calculate arbitrarily many Bäcklund transformations of the initial curve, choosing constants $C_1, C_2, C_3, \ldots$. (Recall also that once a single solution of a Riccati equation is known, all others can be obtained by quadrature.) The superposition principle (4) also makes sense in the limit as $C_2 \to C_1$, if we fix initial values for β_1 and β_2 independent of C_1, C_2.

1.2 The Frenet frame and the linear system

In this section we will see that, if we know the solution of the Frenet equations for a given analytic curvature function $\kappa(s)$, as analytic functions of the constant torsion τ, then we already know the solution to the Riccati equation (1) that we have to solve to compute the Bäcklund transformation for the corresponding constant torsion curves.

For the given curvature function $\kappa(s)$, suppose $\Psi(s; \lambda)$ is the fundamental solution matrix for the linear system

$$\frac{d\psi}{ds} = \frac{1}{2} \begin{pmatrix} \lambda & \kappa(s) \\ -\kappa(s) & -\lambda \end{pmatrix} \psi. \tag{5}$$

(It follows from standard ODE theory that Ψ is an analytic function of λ.) When $\lambda = -i\tau$ for $\tau \in \mathbb{R}$, this system encodes the Frenet equations, in the following way. If

$$\gamma(s) = i \Psi^{-1} \left. \frac{d\Psi}{d\lambda} \right|_{\lambda = -i\tau},$$

then γ is a curve in $su(2)$, which we will identify with $\mathbb{R}^3$ (see below). One can verify that γ satisfies the Frenet equations for curvature $\kappa(s)$ and constant torsion

τ, when the Frenet frame is given by the $su(2)$ matrices

$$\begin{aligned} T &= \frac{1}{2}\Psi^{-1}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\Psi \\ N &= \frac{1}{2}\Psi^{-1}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\Psi \\ B &= \frac{1}{2}\Psi^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\Psi \end{aligned} \tag{6}$$

Now suppose that $\Psi(s;\lambda)$ were extended complex analytically from λ pure imaginary to all $\lambda \in \mathbb{C}$. We observe then that for $\lambda = C \in \mathbb{R}$, the coefficient matrix in (5) is real, and for any real vector solution ψ, the ratio of entries $y = -\psi_1/\psi_2$ satisfies the Riccati equation

$$\frac{dy}{ds} = Cy - \frac{\kappa}{2}(1+y^2).$$

Since this is the same as the equation satisfied by $\tan(\beta/2)$ under the Bäcklund transformation, we obtain a Bäcklund transformation for the curve γ using

$$\sin\beta = \frac{-\psi_1\psi_2}{\psi_1^2+\psi_2^2}, \qquad \cos\beta = \frac{\psi_2^2-\psi_1^2}{\psi_1^2+\psi_2^2}. \tag{7}$$

In practice, we may not know the fundamental matrix Ψ explicitly, but if we know the Frenet frame for the curve, normalized so that (T, N, B) is the identity matrix at $s = 0$, and use the identification

$$(1,0,0) \to \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (0,1,0) \to \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (0,0,1) \to \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of $\mathbb{R}^3$ with $su(2)$, then (6) gives us values for all possible quadratics in the entries of Ψ. Thus, for any linear combination ψ of the columns of Ψ, we can express the quadratics in (7) in terms of analytic extensions of those obtained using (6).

1.3 Torus knots and closed elastic rods of constant torsion

In a recent paper [L-S4], Langer and Singer formulate and solve the Euler-Lagrange equations associated to the *Kirchoff elastic rod*, by which is meant a curve that is critical for some linear combination of $\int \kappa^2 ds$, $\int \tau ds$ and length. Among the solutions they obtain are two families of constant torsion curves. For the family we will use, the curvature is given by an elliptic cosine, and the shape of the rod is governed by the torsion, the maximum curvature κ_0, and the elliptic modulus p:

$$\kappa(s) = \kappa_0 \,\mathrm{cn}\,(x,p), \qquad \text{where } x = \frac{\kappa_0 s}{2p}.$$

(In what follows, we will take $\kappa_0 = 1$.) Up to scale, then, the shape of the rod depends only on two parameters; we will use the parameters p and $\sigma = 2\tau$, and the coordinate x along the rod. We will show how to choose the parameters to obtain closed elastic rods of constant torsion.

Langer and Singer compute the position of the rod in a system of cylindrical coordinates (r, θ, z) that are generated by the Killing fields associated to the rod (see [L-S4], §5). In these coordinates, r is already a $2K$-periodic function of x, where $K = K(p)$ is the complete elliptic integral of the first kind:

$$r = \frac{1}{\mu}\sqrt{\frac{1}{m} - \operatorname{sn}^2 x},$$

where

$$\mu = \frac{1}{4}\sqrt{(p^{-2} - \sigma^2)^2 + 4\sigma^2},$$
$$m = 16\mu^2/(p^{-2} + \sigma^2)^2.$$

Closure in z imposes (cf. equation (26) in [L-S4])

$$\sigma^2 = \frac{1}{p^2}(2\frac{E(p)}{K(p)} - 1), \tag{8}$$

where $E(p)$ is the complete elliptic integral of the second kind. (David Singer has shown that constant torsion rods in the other family, whose curvature is given elliptic dn, are never closed in the z coordinate.) This implies $|\sigma| < p^{-1}$ and that p has a maximum possible value $p_{\max} \approx .9089085$ at which σ^2 approaches zero. (We will take $\sigma > 0$, giving positive torsion curves; curves obtained using $\sigma < 0$ will differ by a reflection.) Using the remaining parameter p, we can attempt to make the change in θ over a $2K$ period equal to a rational multiple of 2π. The Euler-Lagrange equations imply that

$$\frac{d\theta}{dx} = -\frac{p\sigma}{\mu}\left(\lambda_1 + \frac{(p^{-2} - 1)(p^{-2} - \sigma^2)}{2(p^{-2} + \sigma^2)(1 - m\operatorname{sn}^2 x)}\right), \qquad \text{where } \lambda_1 = (\sigma^2 - p^{-2} + 2)/4.$$

(This formula is due to Singer.) So, the change in θ can be expressed in terms of complete elliptic integrals:

$$-\frac{\Delta\theta}{2K} = E(\xi, p') + \left(\frac{E}{K} - 1\right) F(\xi, p') + \frac{\lambda_1 p\sigma}{\mu} =: \Lambda(\sigma, p) \tag{9}$$

where

$$\xi = \pi/2 - 2\arctan\left(\frac{2\sigma}{p^{-2} - \sigma^2 + 4\mu}\right)$$

(cf. formula 413.01 in [B-F]).

Remark. It will be desirable to have a formula for $\Lambda(p,\sigma)$ that does not involve μ as denominator. Using Legendre's relation, and the addition formula and imaginary transformation for Jacobi's zeta function $Z(x)$ (see [W-W], §22.73), we obtain

$$\Lambda = -iZ\left(\mathrm{sn}^{-1}\alpha\right) + p^2\sigma\alpha, \qquad \text{where } \alpha = \frac{4\mu}{p(p^{-2}+\sigma^2)}. \tag{10}$$

The following result now follows directly from the above formulas and those in [L-S4]:

Theorem 1.2. *Given any relatively prime integers m, n such that $|m/n| < 1/2$, there exists a smooth closed elastic rod of constant torsion with the knot type of an (m,n) torus knot.*

Proof. We use continuity, noting that the limiting values of $\Delta\theta/2\pi$ are zero as $p \searrow 0$, and one half as $p \to p_{\max}$, to assert that $\Delta\theta/2\pi = m/n$ for some modulus p. Then one checks that, over the course of one $2K$ period, $r(x)$ and $z(x)$ describe a simple closed curve in the rz plane.□

Remarks. It is interesting that this is exactly the same set of knot types as are available among elastic curves (see [L-S1]), which of course are critical for a combination of $\int \kappa^2 ds$ and length, and do not have constant torsion (unless they are planar). Since a computer-generated plot of $\Delta\theta$ shows it to be a monotone increasing function of p, we expect that, similar to the situation for elastic curves, there is a unique constant torsion elastic rod for each possible torus knot type. (However, by arranging that $\Delta\theta/2\pi = 1/n$ for any $n \geq 3$, we can get many unknots.) Some of these torus knots, as well as some unknots, are shown in Figure 1.2. It is also apparent that, as $p \to p_{\max}$ and the torsion approaches zero, the curve approaches a figure-eight elastic curve contained in a plane through the z-axis.

For the purpose of computing their Bäcklund transformations, we need to compute Frenet frames for these elastic rods. Using the formulas in [L-S4], one can express the Frenet frame in a cylindrical coordinate basis:

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} -\dfrac{\mathrm{cn}\, x \,\mathrm{sn}\, x \,\mathrm{dn}\, x}{2p\mu^2 r^2} & -\dfrac{\sigma(p^{-2}+\sigma^2-4\lambda_1 \mathrm{sn}^2 x)}{8\mu^3 r^2} & \dfrac{\mathrm{cn}^2 x - 2\lambda_1}{2\mu} \\ -\dfrac{\mathrm{cn}\, x(p^{-2}+\sigma^2-2\mathrm{sn}^2 x)}{4\mu^2 r^2} & \dfrac{\lambda_1 \sigma \mathrm{sn}\, x \,\mathrm{dn}\, x}{2p\mu^3 r^2} & -\dfrac{\mathrm{sn}\, x \,\mathrm{dn}\, x}{2p\mu} \\ \dfrac{\sigma \mathrm{sn}\, x\,\mathrm{dn}\, x}{2p\mu^2 r^2} & -\dfrac{(p^{-4}-\sigma^4)\mathrm{cn}\, x}{16\mu^3 r^2} & -\dfrac{\sigma \mathrm{cn}\, x}{2\mu} \end{pmatrix} \begin{pmatrix} r\partial_r \\ \partial_\theta \\ \partial_z \end{pmatrix}$$

To convert to Cartesian coordinates, we need the cylindrical coordinates r, θ as functions along the elastic rod. Using formulas that express incomplete elliptic

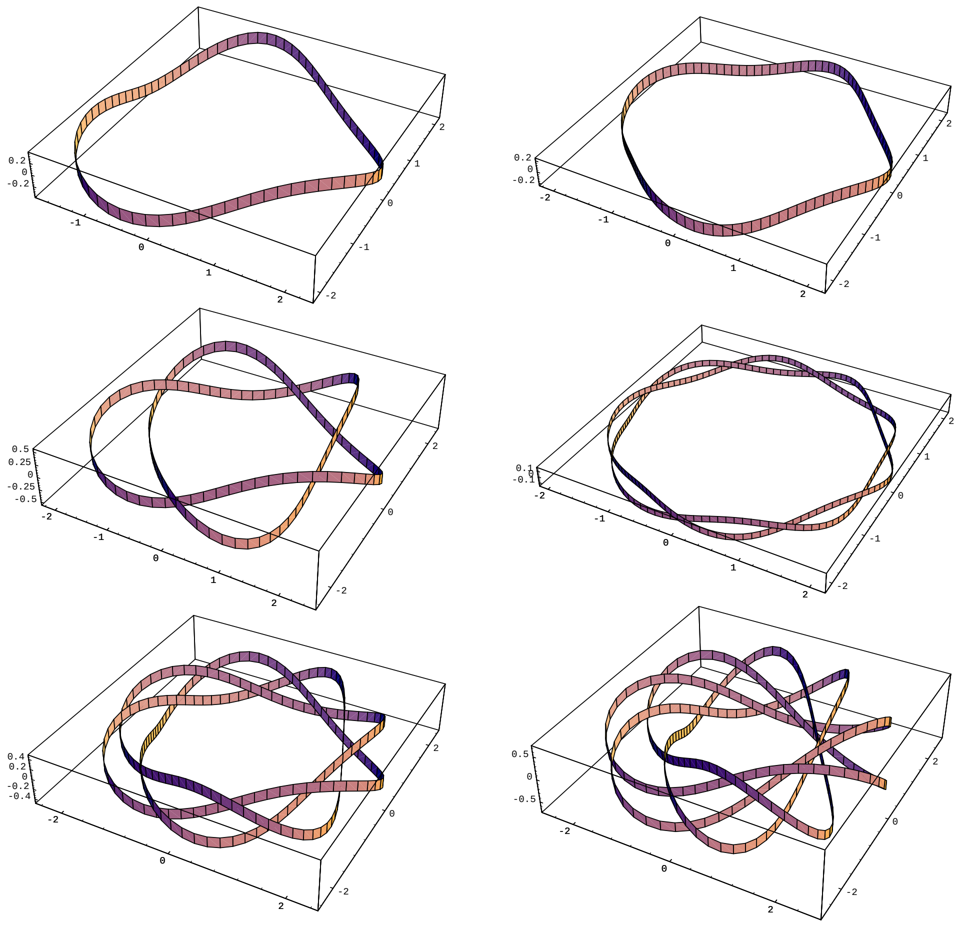

FIGURE 1.2. Some closed elastic rods of constant torsion. (The curves here have been thickened by a vertical ribbon.)

integrals of the third kind in terms of theta functions (cf. formula 434.01 in [B-F]), we get

$$\begin{aligned} r\cos\theta &= \sqrt{\frac{2Kp'}{\pi p}}\;\frac{e^{-i\Lambda x}\Theta_1(x-i\widehat{F})+e^{i\Lambda x}\Theta_1(x+i\widehat{F})}{2\mu\Theta(x)H_1(i\widehat{F})} \\ r\sin\theta &= \sqrt{\frac{2Kp'}{\pi p}}\;\frac{e^{-i\Lambda x}\Theta_1(x-i\widehat{F})-e^{i\Lambda x}\Theta_1(x+i\widehat{F})}{2\mu i\Theta(x)H_1(i\widehat{F})} \end{aligned} \tag{11}$$

where $\widehat{F} = F(\xi, p')$ and $\Lambda = \Lambda(\sigma, p)$ is defined in (9).

Now let H be the matrix whose rows are T, N, B. Let $H_0 = H|_{x=0}$, and let $G(x,\sigma) = H_0^{-1}H$; G is the normalized Frenet frame. The above formulas can be used to express the entries of G as complex analytic functions of σ in a disk about the origin in the σ-plane. (One can also desingularize the formulas near the points $\sigma = \pm i \pm p'/p$, where $\mu = 0$.) In particular, from $G(x, 2iC)$ we may, using the formulas (7) in the previous section, construct the Bäcklund transformation of an elastic rod of constant torsion. However, the new curve is not necessarily closed;

in subsequent sections we will see how to obtain closure and how severely closure restricts the shape of the new curve.

Remarks.

1. From the formulas in [L-S4] one can also obtain

$$z(x) = \frac{1}{\mu p}\frac{\Theta'(x)}{\Theta(x)}. \tag{12}$$

This bears a striking resemblance to the formula obtained by Mumford [M] for planar elastic curves in terms of theta functions. In fact, one can check that as $p \to p_{\max}$ and our elastic rod becomes planar, (11) and (12) agree with Mumford's formula in the limit.

2. One can derive formulas quite similar to (11) and (12) for elastic rods in general.

1.4 Closure of the Bäcklund transformation

Suppose we have generated a Bäcklund transformation of a constant torsion curve γ, by obtaining the fundamental matrix Ψ for the linear system (5), selecting a non-zero vector solution

$$\psi = \Psi \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad c_1, c_2 \in \mathbb{R},$$

and performing the transformation using (7) and (2). Suppose that γ is closed and has length L. Then the Bäcklund transform $\widetilde{\gamma}$ will be closed if the vector $V = \cos\beta T + \sin\beta N$ has period kL for k a positive integer.

Proposition 1.3. *Suppose V has period kL. If k is even or γ is even, then (c_1, c_2) is an eigenvector of $\Psi|_{s=kL}$. If k is odd and γ is odd, then (c_1, c_2) is an eigenvector of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi|_{s=kL}$.*

When γ is an elastic rod of constant torsion, we may use the entries of the normalized Frenet frame $G(x, \sigma)$, together with the formulae (6), to obtain all quadratics in the entries of Ψ. In particular, to obtain the eigenvectors required in the above proposition, we may use the squares of the relevant matrices.

Suppose that the elastic rod closes up after n periods of length $2K$ in the x parameter. (The rod is even or odd exactly as n is even or odd.) If kn is even, then we calculate that

$$\Psi^2|_{x=2knK} = \cos(x\Lambda)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \sin(x\Lambda)\begin{pmatrix} i\dfrac{1-2\lambda_1}{2\mu} & \dfrac{\sigma}{2\mu} \\ -\dfrac{\sigma}{2\mu} & -i\dfrac{1-2\lambda_1}{2\mu} \end{pmatrix}$$

where Λ is given by (9). Note that, when we want to calculate Bäcklund transformations, we set $\sigma = 2\tau = 2iC$ for $C \in \mathbb{R}$, $C \neq 0$, and then Λ is purely imaginary and Ψ^2 is real. Moreover, the eigenvalues $\cos(x\Lambda) \pm i\sin(x\Lambda)$ of this matrix are always distinct. If kn is odd, then

$$\left(\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\Psi\right)^2|_{x=2knK} = -\cos(x\Lambda)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sin(x\Lambda)\begin{pmatrix} i\dfrac{1-2\lambda_1}{2\mu} & \dfrac{\sigma}{2\mu} \\ -\dfrac{\sigma}{2\mu} & -i\dfrac{1-2\lambda_1}{2\mu}\end{pmatrix}$$

—i.e., the same matrix as above, up to a minus sign. Thus, in either case the eigenvalues are distinct. The corresponding eigenvectors are

$$v_+ = \begin{pmatrix} -\dfrac{\sigma}{2\mu} \\ i\left(1+\dfrac{1-2\lambda_1}{2\mu}\right)\end{pmatrix}, \qquad v_- = \begin{pmatrix} i\left(1+\dfrac{1-2\lambda_1}{2\mu}\right) \\ -\dfrac{\sigma}{2\mu}\end{pmatrix}. \tag{13}$$

Since these eigenvectors are independent of k, we conclude that *when γ is a closed elastic rod of constant torsion, $\widetilde{\gamma}$ closes up after k circuits around γ if and only if it closes up after one circuit.*

1.5 Elastic rods again

In this section we will see that, for any value of the Bäcklund parameter C, a closed Bäcklund transform of a closed elastic rod of constant torsion is congruent to that rod. Since the two curves have the same torsion, it will suffice to show that they have the same curvature function, up to a phase shift.

Let $q = 2C$ and $\sigma = iq$ for $q \in \mathbb{R}$. Then by (7) the initial values for β corresponding to eigenvectors v_+ and v_- given above are

$$\sin\beta = \frac{2q}{p^{-2}+q^2}, \qquad \cos\beta = \pm\frac{\sqrt{(p^{-2}+q^2)^2-4q^2}}{p^{-2}+q^2}. \tag{14}$$

Since $\kappa = \operatorname{cn} x$, the differential equation for β is

$$d\beta/dx = p(q\sin\beta - 2\operatorname{cn} x). \tag{15}$$

If $\widetilde{\kappa}$ were $\operatorname{cn}(x-a)$ then (3) would imply that

$$q\sin\beta = \operatorname{cn} x - \operatorname{cn}(x-a) \tag{16}$$

and

$$d\beta/dx = -p(\operatorname{cn} x + \operatorname{cn}(x-a)).$$

Then by integration,

$$\beta = -[\sin^{-1}(p\operatorname{sn} x) + \sin^{-1}(p\operatorname{sn}(x-a))] + C_1 \tag{17}$$

for some constant C_1, giving

$$\begin{aligned}\sin(\beta - C_1) &= -p[\operatorname{sn} x \operatorname{dn}(x-a) + \operatorname{dn} x \operatorname{sn}(x-a)]\\ \cos(\beta - C_1) &= \operatorname{dn} x \operatorname{dn}(x-a) - p^2 \operatorname{sn} x \operatorname{sn}(x-a).\end{aligned}$$

We can find the constants a and C_1 by setting $x = 0$: (16) gives

$$\operatorname{cn} a = (p^{-2} - q^2)/(p^{-2} + q^2)$$

and so

$$\operatorname{sn} a = \pm \frac{2p^{-1}q}{p^{-2} + q^2}, \tag{18}$$

while (17) gives $\sin(\beta - C_1) = p \operatorname{sn} a$ when $x = 0$. Comparing with (14) shows that we should use $C_1 = 0$ or $C_1 = \pi$; since $\cos(\beta - C_1)$ is always positive in (17), we use $C_1 = 0$ and the plus sign in (18) for v_+, $C_1 = \pi$ and the minus sign in (18) for v_-.

It remains only to be seen that (17) gives a solution to (15). (We will show this for the initial values associated to v_+, using $C_1 = 0$ and the plus sign in (18); the v_- case is similar.) Let $b = a/2$. Then from (17),

$$d\beta/dx = -2p \operatorname{cn} x - 2p \left(\frac{\operatorname{sn} b \operatorname{dn} b \operatorname{sn}(x-b) \operatorname{dn}(x-b)}{1 - p^2 \operatorname{sn}^2 b \operatorname{sn}^2(x-b)} \right)$$

(cf. 123.03 in [B-F]). Since

$$pq \sin\beta = -2p^2 q \frac{\operatorname{sn}(x-b) \operatorname{dn}(x-b) \operatorname{cn} b}{1 - p^2 \operatorname{sn}^2 b \operatorname{sn}^2(x-b)},$$

we will be done if we can show that $pq \operatorname{cn} b = \operatorname{sn} b \operatorname{dn} b$. This now follows from the double angle formula

$$\frac{\operatorname{sn} a}{1 + \operatorname{cn} a} = \frac{\operatorname{sn} b \operatorname{dn} b}{\operatorname{cn} b}.$$

Now, by construction, $\widetilde{\kappa} = \operatorname{cn}(x-a)$ and hence $\widetilde{\gamma}$ is congruent to the elastic rod γ.

1.6 Linking numbers

In this section we use the explicit solution of (1) obtained for an elastic rod γ in the previous section, to compute the linking number of γ and its twin $\widetilde{\gamma}$ in terms of the *self-linking* of γ (see [Po1] for discussion of self-linking).

Theorem 1.4. *If $|C|$ is sufficiently small,*

$$Lk(\gamma, \widetilde{\gamma}) = SL(\gamma) - n/2 \tag{19}$$

where n is the number of $2K$ periods of γ.

The proof will be an application of White's formula (see [W], [Po2]). Before proving this result, we should remind the reader that the self-linking number $SL(\gamma)$ can be thought of as the linking number of γ with the curve traced out by the endpoint of the Frenet normal N (assuming γ is even), suitably scaled so that the vector never intersects other points of γ. As such, $SL(\gamma)$ is the sum of a contribution from the twisting of N about γ and a contribution from the twisting of γ about itself—a measure of the knottedness of γ. In (19), the former contribution is cancelled out by the $-n/2$. In particular, when γ is unknotted (as it is when, for example, we arrange that $\Delta\theta = \pi/2$), then $Lk(\gamma, \widetilde{\gamma}) = 0$.

Proof. Let us recall White's Formula. Let γ be a smooth closed space curve, oriented by unit tangent vector T, and V a unit normal vector along γ. Let γ_{V} be the curve traced out by the endpoints of δV, with the obvious orientation, where $\delta > 0$ is chosen small enough that the ribbon spanned by δV, with boundary $\gamma \bigcup \gamma_{\mathrm{V}}$, is embedded. Then

$$Lk(\gamma, \gamma_{\mathrm{V}}) = Wr(\gamma) + \frac{1}{2\pi} \int (T \times V) \cdot dV, \tag{20}$$

where the *writhe* is given by the integral

$$Wr(\gamma) = \int\!\!\int_{\gamma\times\gamma} e^* dS,$$

wherein for $(x, y) \in \gamma \times \gamma$, $x \neq y$, e is the unit vector from x to y, and dS is the element of area on the unit sphere. Before White, Pohl [Po1] obtained a special case of this formula,

$$SL(\gamma) = Wr(\gamma) + \frac{1}{2\pi} \int \tau ds, \tag{21}$$

which allows us to express the writhe in terms of the self-linking of γ. (While Pohl did not define the self-linking number for curves with inflection points, we will take (21) as defining $SL(\gamma)$ for such curves. If γ is an even curve, this $SL(\gamma)$ is indeed the linking number of γ and γ_N—for either choice of N—while if γ is odd, $SL(\gamma)$ is a half-integer.)

We next remark that it is easy to extend (20) to the case where V is a transverse vector field along γ:

$$Lk(\gamma, \gamma_{\mathrm{V}}) = Wr(\gamma) + \frac{1}{2\pi} \int (T \times U) \cdot dU, \tag{22}$$

where U is the unit vector in the direction of the orthogonal projection of V into the normal plane along γ. However, for the Bäcklund transformation we have

$$V = \cos\beta\, T + \sin\beta\, N,$$

which fails to be transverse to γ whenever $\sin\beta = 0$. In order to use (22) to calculate the linking number, we will take a small perturbation $\widetilde{V}$ of V that is transverse to γ, also ensuring that the ribbon spanned by $\widetilde{V}$ is embedded. Then, since, as shown by the Gauss integral formula, the linking number depends continously on the two curves, our calculation of $Lk(\gamma, \gamma_{\widetilde{V}})$ will give $Lk(\gamma, \gamma_{\mathrm{V}})$ also.

We may assume that $\kappa \neq 0$ and $\tau \neq 0$ in the vicinity of any point where V is tangent to γ. A local calculation then shows that, when V is a positive multiple of T at the point of tangency, the perturbation

$$\widetilde{V} = V - \epsilon\kappa\tau B, \qquad \epsilon > 0 \tag{23}$$

makes the ribbon spanned by $\delta\widetilde{V}$ embedded, for δ and ϵ sufficiently small. (It is interesting to note that if ϵ is negative in (23), then no matter how small δ is chosen, the ribbon fails to be embedded near the point of tangency. The reader may visualize this by imagining a momentarily tangent vector field V along a right-handed helix, with $\kappa > 0$ and $\tau > 0$: the vector field must be perturbed down, not up, in order to prevent the perturbed vector field from intersecting the helix.) Now (22) gives

$$Lk(\gamma, \gamma_{\mathrm{V}}) = SL(\gamma) + \frac{1}{2\pi}\int d\arctan\left(\frac{\sin\beta}{\epsilon\kappa\tau}\right)$$

When γ is a constant torsion elastic rod, and the Bäcklund transform is constructed using eigenvector v_+,

$$\sin\beta = \frac{\mathrm{cn}\,x - \mathrm{cn}\,(x-a)}{q}, \qquad \mathrm{sn}\,a = \frac{2pq}{1+p^2q^2}$$

and $\cos\beta > 0$ always, showing that our assumptions hold. Furthermore, $\sin\beta/\kappa$ goes from $+\infty$ to $-\infty$ between inflection points, and hence

$$Lk(\gamma, \gamma_{\mathrm{V}}) = SL(\gamma) - n/2.$$

The same result holds when we use eigenvector v_-, but since $\cos\beta < 0$ we have to use $\epsilon < 0$ in (23); however, $\mathrm{sn}\,a = -\dfrac{2pq}{1+p^2q^2}$ makes up for the change in sign. $\square$

1.7 Bäcklund transformations and the LIE hierarchy

The constant torsion elastic rods are among the "soliton" curves for the Localized Induction Equation (LIE), and as such are critical for some linear combination of the integral invariants associated with the LIE. In this section, we will formulate a precise conjecture as to how the Bäcklund tranformation takes solitons to solitons, and in particular how the coefficients in the linear combination of invariants change. Ercolani, Forest and Mclaughlin [E-F-M] use arguments from algebraic geometry to address the analogous question for n-soliton solutions of the sine-Gordon equation. They find that in general a Bäcklund transformation (3) takes n-solitons to (n+1)-solitons; what we describe below is a more geometric argument supporting the same conclusion. In the particular case when the initial condition of the Riccati equation selects one eigenvector of the transfer matrix of the linear system, producing closed curves as we discussed above, then [E-F-M] find that the Bäcklund formula takes an n-soliton to another n-soliton; this of course agrees with our results on closed Bäcklund transformations of elastic rods.

We begin with the observation, due to Langer and Perline [Pe], that along a curve of constant torsion τ the linear combinations of the LIE vector fields $\{X_k\}_{k=0}^{\infty}$ defined by

$$B_n = \sum_{k=0}^{2n} \binom{2n}{k} (-\tau)^{2n-k} X_{k+1}, \qquad n \geq 0$$

are purely binormal and do not involve τ. For example,

$$B_0 = X_1 = \kappa B, \qquad B_1 = X_3 - 2\tau X_2 + \tau^2 X_1 = -(\kappa'' + k^3/2)B.$$

A *planar-like n-soliton* [Pe] is a curve of constant torsion along which some constant-coefficient linear combination $\sum_{k=0}^{n} a_k B_k$ vanishes. For example, a helix is a planar-like 1-soliton. Because the LIE vector fields X_k are Hamiltonian vector fields associated with certain densities involving κ, τ and their derivatives (see [L-P1] for details), these solitons are critical curves for a natural variational problem. For example, since X_1, X_2, X_3 are Hamiltonian vector fields for $\int ds$, $\int \tau ds$, $\int \frac{1}{2}k^2 ds$, an elastic rod of constant torsion is also a planar-like 1-soliton.

It is easy to show, using the LIE recursion operator [L-P2], that if along a constant torsion curve γ we define

$$U_n = \sum_{k=0}^{2n} \binom{2n}{k} (-\tau)^{2n-k} X_k, \tag{24}$$

then $\sum_{k=0}^{n} a_k B_k = 0$ if and only if

$$J = \sum_{k=0}^{n} a_k U_k \tag{25}$$

is a constant vector. We call J a *Killing field along* γ because it is the restriction of a vector field generating a symmetry of Euclidean space, in this case translation. Combinations of LIE vector fields that are Killing fields play an important role in showing that variational problems associated to combinations of the LIE functionals yield finite-dimensional completely integrable Hamiltonian systems (see [L-S2], [L-S3], [L-S4]). So, it is important to see how Killing fields along planar-like n-solitons interact with the Bäcklund transformation.

Conjecture 1.5. *Let γ be a constant torsion n-soliton, and let $\widetilde{\gamma}$ be a Bäcklund transformation of γ. Let $\overline{U}_k$ denote the combinations of LIE vectors defined by (24), computed along $\widetilde{\gamma}$. Then if J is the constant vector defined by (25) along γ,*

$$J = \frac{1}{\tau^2 + C^2}\left[\sum_{k=0}^{n} a_k \overline{U}_{k+1} + b\overline{U}_0 + C^2 \sum_{k=0}^{n} a_k \overline{U}_k\right]$$

where C is the parameter in (1), (2) and b is a constant depending only on γ.

For any given n, one can mechanically verify this conjecture using the formulae for the vector fields U_k, $\overline{U}_k$, the equation (3) for the curvature of $\widetilde{\gamma}$, the differential equation (1), and the ODE for the curvature of γ implied by $\sum_{k=0}^{n} a_k B_k = 0$. (The constant b arises as a first integral of this equation.)

We have verified the conjecture for $n = 0, 1, 2$; hence we know that the Bäcklund transform takes a planar-like 1-soliton to a planar-like 2-soliton, and takes a planar-like 2-soliton to a planar-like 3-soliton. However, the results of previous sections show that a *closed* transform of a closed 1-soliton is still a 1-soliton; an intuitive explanation for this can be given for small values of the arbitrary constant C in the Bäcklund formula. When $C \to 0$, since $\sin\beta$ is bounded, the Riccati equation becomes approximately $\dfrac{d\beta}{ds} \simeq -\kappa$. Therefore $\beta(s) \simeq -\theta(s) = -\int^s \kappa(s')ds'$ and, to first order,

$$\tilde{\gamma} = \gamma + \frac{2C}{\tau^2}\left[\cos\theta T - \sin\theta N\right].$$

In other words, the Bäcklund transformation flows along the vector field

$$\gamma_t \simeq W = \cos\theta T - \sin\theta N.$$

This vector field W, discussed in [L-P2], is known as the *trigonometric vector field*, and the associated evolution for $\theta = \int^s \kappa(s')ds'$ is described by the sine-Gordon equation, which *preserves* soliton type. Moreover, for periodic solutions of the sine-Gordon equation, the limiting Bäcklund formula described above transforms closed n-soliton curves into closed n-soliton curves.

In the second part of this paper, we will perform iterated Bäcklund transformations on the 1-solitons to obtain new closed curves that are multi-solitons for the LIE hierarchy.

II. The double Bäcklund transformation

In Part I we used Bäcklund transformations to produce new constant torsion elastic rods from given ones. Moreover, we verified that imposing the closure condition on the transformed curves restricts them to the same congruence class as the original elastic rod.

More exotic and interesting constant torsion curves are produced by successive iterations of the Bäcklund transformation starting with a given elastic rod of constant torsion.

2.1 A gauge formula for the Bäcklund transformation

We present an alternative derivation of the Bäcklund formula based on a gauge transformation for the associated linear system, as described in [E-F-M]. This approach is most convenient for constructing iterated Bäcklund transformations, since it reduces each step of the iteration to a purely algebraic computation. We will show that the Riccati equation (equivalently, the linear system) needs to be solved only once at the initial step of the iteration.

We begin with the following form of the associated linear system

$$\frac{d\phi}{ds} = \frac{1}{2}\begin{pmatrix} i\kappa(s) & \lambda \\ \lambda & -i\kappa(s) \end{pmatrix}\phi, \tag{26}$$

which simplifies the formulation of the Bäcklund transformation. We observe that equation (26) is transformed into the linear system (5) by setting $\phi = A\psi$, where $A = \frac{1}{2}\begin{pmatrix} 1-i & -1-i \\ 1-i & 1+i \end{pmatrix}$.

Using the fundamental solution matrix Φ of (26), we identify the curve of constant torsion τ with the $su(2)$ matrix

$$\gamma(s) = i\Phi^{-1}\left.\frac{d\Phi}{d\lambda}\right|_{\lambda=-i\tau}, \tag{27}$$

and compute its Frenet frame

$$\begin{aligned} T &= \frac{1}{2}\Phi^{-1}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\Phi \\ N &= \frac{1}{2}\Phi^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Phi \\ B &= \frac{1}{2}\Phi^{-1}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\Phi. \end{aligned} \tag{28}$$

The Bäcklund formula is a consequence of the following

Proposition 2.1. *Let $\chi(s;\nu)$ be a solution of the linear system (26) for $\lambda = \nu$, and let $\phi(s;\lambda)$ be a solution of (26) for an arbitrary $\lambda \in \mathbb{C}$. Then, the gauge transformation*

$$\phi^{(1)}(s;\lambda) = \begin{pmatrix} \lambda & -\nu\dfrac{\chi_1}{\chi_2} \\ -\nu\dfrac{\chi_2}{\chi_1} & \lambda \end{pmatrix}\phi(s;\lambda), \tag{29}$$

produces a solution $\phi^{(1)}$ of

$$\frac{d\phi}{ds} = \frac{1}{2}\begin{pmatrix} i\kappa^{(1)}(s) & \lambda \\ \lambda & -i\kappa^{(1)}(s) \end{pmatrix}\phi, \tag{30}$$

where

$$\kappa^{(1)}(s) = \kappa(s) + i\nu\left(\frac{\chi_1}{\chi_2} - \frac{\chi_2}{\chi_1}\right). \tag{31}$$

The proof is a direct verification of formulas (29) and (31).

We will note for future reference that $z = \chi_1/\chi_2$ satisfies the Riccati equation $dz/ds = i\kappa z + \nu(1-z^2)/2$.

The gauge transformation (29) can be normalized to produce the fundamental matrix for (30):

$$\Phi^{(1)}(s;\lambda) = \frac{1}{\sqrt{\lambda^2-\nu^2}}\begin{pmatrix} \lambda & -\nu z \\ -\nu/z & \lambda \end{pmatrix}\Phi(s;\lambda). \tag{32}$$

Assume that $\kappa(s)$ is real, and suppose for the moment that $\lambda = -i\tau$ for $\tau \in \mathbb{R}$. The transformation (31) does not, in general, produce a real-valued function $\kappa^{(1)}(s)$ unless some condition is imposed on ν. Note that the matrix Φ now takes values in $SU(2)$, and (32) becomes

$$\Phi^{(1)}(s;-i\tau) = \frac{1}{\sqrt{\tau^2+\nu^2}}\begin{pmatrix} \tau & -i\nu z \\ -i\nu/z & \tau \end{pmatrix}\Phi(s;-i\tau).$$

This gauge transformation is in $SU(2)$ only if $\bar{\nu}\bar{z} = \nu/z$, which implies that ν is real and z has unit modulus. In this case, $\kappa^{(1)}$ is real. Using the reconstruction formula

$$\tilde{\gamma}(s) = i\Phi^{(1)^{-1}}\left.\frac{d\Phi^{(1)}}{d\lambda}\right|_{\lambda=-i\tau},$$

we obtain the following expression for the transformed curve:

$$\tilde{\gamma}(s) = \gamma(s) + \frac{i\nu}{\nu^2+\tau^2}\Phi^{-1}\begin{pmatrix} 0 & z \\ 1/z & 0 \end{pmatrix}\Phi. \tag{33}$$

If we make the substitution $z = -e^{-i\beta}$, then $\beta(s)$ satisfies the equation (1) for the Bäcklund transformation, with $\nu = C$, and (33) becomes (2), in terms of the Frenet frame for γ given by (28).

2.2 The iterated formula

The gauge transformation (29) can be iterated algebraically once the solution of the linear system (26) at an initial curvature function $\kappa(s)$ is known. In fact, formula (29) produces the eigenfunction of (30) out of which the new gauge matrix is constructed.

We begin with some symmetry considerations. Given a solution $\chi(s,\nu)$ of the linear system (26) at a pair $(\kappa(s),\nu)$, then

$$\chi(s,\bar{\nu}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\overline{\chi(s,\nu)}. \tag{34}$$

is a solution of (26) at $(\kappa(s), \bar\nu)$. We use this property to describe the following iterated Bäcklund transformation

$$\gamma \xrightarrow{\nu} \gamma^{(1)} \xrightarrow{\bar\nu} \gamma^{(2)}.$$

(The use of the symmetry (34) is crucial in constructing real-valued curvature functions and curves in $su(2)$.) The following lemma is obtained by direct computation:

Lemma 2.2. *Let $f(s;\bar\nu)$ be a solution of (26) at $(\kappa(s), \bar\nu)$, and let*

$$\zeta(s;\bar\nu) = \begin{pmatrix} \bar\nu & -\nu\dfrac{\chi_1}{\chi_2} \\ -\nu\dfrac{\chi_2}{\chi_1} & \bar\nu \end{pmatrix} f(s;\bar\nu)$$

be the new solution at $(\kappa^1(s), \bar\nu)$ constructed by means of the gauge transformation (29) . Then, two applications of the gauge formula (29), using solutions $\chi(s;\nu)$ and $\zeta(s;\bar\nu)$, gives

$$\phi^{(2)}(s;\lambda) = \mathcal{T}(\lambda,\nu)\phi(s;\lambda)$$

$$:= \frac{1}{\sqrt{(\lambda^2-\nu^2)(\lambda^2-\bar\nu^2)}} \begin{pmatrix} \lambda^2 + |\nu|^2\dfrac{\chi_2\zeta_1}{\chi_1\zeta_2} & -\lambda(\nu\dfrac{\chi_1}{\chi_2} + \bar\nu\dfrac{\zeta_1}{\zeta_2}) \\ -\lambda(\nu\dfrac{\chi_2}{\chi_1} + \bar\nu\dfrac{\zeta_2}{\zeta_1}) & \lambda^2 + |\nu|^2\dfrac{\chi_1\zeta_2}{\chi_2\zeta_1} \end{pmatrix} \phi(s;\lambda). \tag{35}$$

Moreover, the matrix $\mathcal{T}(-i\tau,\nu)$, $\tau \in \mathbb{R}$, is unitary if and only if we choose $f(s,\bar\nu) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{\chi(s,\nu)}$.

We can now derive the explicit formula for the transformed curve and for its corresponding curvature function by using the reconstruction formula (27) and the form (35) of the new eigenfunction. We set $\chi_1/\chi_2 = \rho e^{-i\beta}$ (no longer unit modulus), $\lambda = -i\tau$ and we introduce the constant $\alpha = i\dfrac{\nu^2-\bar\nu^2}{(\tau^2+\nu^2)(\tau^2+\bar\nu^2)}$.

Proposition 2.3. *The iterated Bäcklund transformation of the curve $\gamma(s)$ at the eigenvalues ν and $\bar\nu$ is the constant torsion curve*

$$\gamma^{(2)}(s) = \gamma(s) + \alpha\left(pT + qN + rB\right), \tag{36}$$

where

$$p = \frac{-1}{(\nu-\bar\nu\rho^2)(\bar\nu-\nu\rho^2)}\left[\frac{\nu-\bar\nu}{2i}(\tau^2+|\nu|^2)(1+\rho^2)\rho\cos\beta + \frac{\nu+\bar\nu}{2}(\tau^2-|\nu|^2)(1-\rho^2)\rho\sin\beta\right]$$

$$q = \frac{1}{(\nu-\bar\nu\rho^2)(\bar\nu-\nu\rho^2)}\left[\frac{\nu+\bar\nu}{2}(\tau^2-|\nu|^2)(1-\rho^2)\rho\cos\beta - \frac{\nu-\bar\nu}{2i}(\tau^2+|\nu|^2)(1+\rho^2)\rho\sin\beta\right]$$

$$r = \tau|\nu|^2\frac{\rho^4-1}{(\nu-\bar\nu\rho^2)(\bar\nu-\nu\rho^2)}.$$

The associated curvature function is

$$\kappa^{(2)}(s) = \kappa(s) + \frac{2i\nu(\bar\nu^2 - \nu^2)\rho}{(\nu - \bar\nu\rho^2)(\bar\nu - \nu\rho^2)}\left[(\nu + \bar\nu)(\rho^2 - 1)\cos\beta + i(\bar\nu - \nu)(\rho^2 + 1)\sin\beta\right]. \tag{37}$$

Remarks.

1. We observe that the expression for $\gamma^{(2)} - \gamma$ now contains a binormal component proportional to the torsion of the original curve.

2. The double Bäcklund transformation makes sense for all $\nu \in \mathbb{C}$ and becomes the identity when ν is real. The formula for the new curve depends in general on two complex parameters: ν, and the parameter $c = c_+/c_-$, assuming $\chi = c_+\chi^+ + c_-\chi^-$ is expressed on fixed basis (χ^+, χ^-) of solutions of the linear system at (κ, ν).

2.3 Closure conditions

When discussing the conditions for which the curve produced by a double Bäcklund transformation (36) is closed, we distinguish two cases:

Case (A) is analogous to the situation discussed in §1.5 for a single Bäcklund transformation. For an initial curve γ of length L, $\chi(s;\nu)$ is taken to be an eigenfunction of the *transfer matrix* $\Phi(kL,\nu)$ across k periods of γ. (However, when γ is odd and k is odd, the appropriate transfer matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Phi(kL,\nu)$ instead.) Then formula (36) produces a family of curves of period kL parametrized by $\nu \in \mathbb{C}$.

However, if the initial curve is a closed elastic rod of constant torsion, then one can show directly that, for any $\nu \in \mathbb{C}$, the new curve has length L and is congruent to the original elastic rod. To see this, first note that the Riccati equation for z is now

$$\frac{1}{2p}\frac{dz}{dx} = iz\,\mathrm{cn}\,x + \frac{\nu}{2}(1 - z^2). \tag{38}$$

Since, when $\kappa = \mathrm{cn}\,x$, $\Psi(kL; -i\sigma/2)$ has eigenvectors given by (13) when $\sigma \in \mathbb{R}$, similar formulas hold for the eigenvectors of $\Phi(kL;\nu)$:

$$v_+ = A\begin{pmatrix} -\dfrac{\sigma}{2\mu} \\ i\left(1 + \dfrac{1 - 2\lambda_1}{2\mu}\right)\end{pmatrix}, \qquad v_- = A\begin{pmatrix} i\left(1 + \dfrac{1 - 2\lambda_1}{2\mu}\right) \\ -\dfrac{\sigma}{2\mu}\end{pmatrix}, \tag{39}$$

where, now, $\sigma = 2i\nu$ for arbitrary $\nu \in \mathbb{C}$, and λ_1 and μ are defined in terms of σ as in §1.3. We again observe that, since the eigenvectors are independent of k, the Backlund transformation has period kL if and only if it has period L. We will consider the initial value problem for (38) corresponding to v_+, the other being similar; the initial value for z is then

$$z(0) = \frac{2\sigma - 4\mu}{p^{-2} + \sigma^2},$$

which can be rewritten as

$$z(0) = ip\mathrm{sn}\, a - \mathrm{dn}\, a, \qquad \text{where } \frac{\mathrm{sn}\, a}{1+\mathrm{cn}\, a} = 2p\nu. \tag{40}$$

Proposition 2.4. *Let $p \in (0,1)$ be the elliptic modulus, and let a lie in an open disk about the origin in the complex plane such that $\mathrm{cn}\,(a) \neq -1$. The solution of the initial value problem (38), (40) is given by*

$$z_+ = -(\mathrm{dn}\, x + ip\mathrm{sn}\, x)\,(\mathrm{dn}\,(x-a) + ip\mathrm{sn}\,(x-a))\,.$$

Moreover, $z_+(x)$ satisfies the identity

$$\mathrm{cn}\, x + i\nu\left(z_+ - \frac{1}{z_+}\right) = \mathrm{cn}\,(x-a). \tag{41}$$

Proof. For a, x real, z must be of unit modulus, and the substitution $z = -e^{-i\beta}$ yields the initial value problem (14),(15) solved in §1.5. The solution is

$$\beta_+(x) = -[\sin^{-1}(p\,\mathrm{sn}\, x) + \sin^{-1}(p\,\mathrm{sn}\,(x-a))]$$

and $z_+(x) = -e^{-i\beta_+}$ has the above form and satisfies the required identity. The rest follows by analyticity in a and x.

After a single iteration of the Bäcklund transformation, we have $\kappa^{(1)} = \mathrm{cn}\,(x-a)$ by the above identity. Let $y = x - a$. After the double Bäcklund transformation, we have

$$\kappa^{(2)} = \kappa^{(1)} + i\bar{\nu}(w - 1/w),$$

where w is a solution of

$$\frac{1}{2p}\frac{dw}{dx} = iz\mathrm{cn}\, y + \frac{\bar{\nu}}{2}(1 - z^2).$$

The formulas in §2.2 show that, in performing the second Bäcklund transformation, we use the solution $w_+ = z_+\left(\dfrac{\bar{\nu} - \nu|z_+|^2}{\bar{\nu}|z_+|^2 - \nu}\right)$. We also know from Prop. 2.4 (with x changed to y and ν to $\bar{\nu}$) that

$$w = -(\mathrm{dn}\, y + ip\mathrm{sn}\, y)\,(\mathrm{dn}\,(y-\bar{a}) + ip\mathrm{sn}\,(y-\bar{a}))$$

is a solution. In fact, it can easily be verified, for example by evaluated at $x = 0$, that w and w_+ are the same. Now it follows by the identity (41) that $\kappa^{(2)} = \mathrm{cn}\,(x - a - \bar{a})$, and the new curve is congruent to the original elastic rod.

Case (B). When the transfer matrix is a multiple of the identity matrix, $\chi(s,\nu)$ can be taken to be an arbitrary complex linear combination $c_+\chi^+(s,\nu) + c_-\chi^-(s,\nu)$

of solutions to (26). To see when this is possible, let $\Delta(kL,\nu)$ be the trace of the transfer matrix; in Floquet theory, this is known as the Floquet discriminant associated to the linear system.

Assume for the moment that k is even or γ is even. Since $\det \Phi(kL,\nu) = 1$, the eigenvalues of the transfer matrix will be both 1 or both -1 exactly when $\Delta^2 = 4$. Moreover, if ν is a root of multiplicity two for the function $\Delta^2 - 4$, then the corresponding eigenvectors will be linearly independent. Taking into account the obvious changes for the case when k is odd and γ is odd, we summarize the discussion in the following

Proposition 2.5. *Let $\tilde{\nu}$ be a complex double root of the equation (a) $\Delta^2(kL,\nu) - 4 = 0$, or (b) $\Delta^2(kL,\nu)+4 = 0$ if γ is odd and k is odd. Let $\chi(s) = c_+\chi^+(s;\tilde{\nu}) + c_-\chi^-(s;\tilde{\nu})$, where χ^+, χ^- are the columns of the fundamental matrix of (26). Then, formula (36) produces a family of closed curves of period nL, parametrized by $\omega = c_+/c_- \in \mathbb{C}$.*

We will loosely refer to the new closed curves as k-fold covers of the original curve.

We should note that, when the initial curve is an elastic rod of constant torsion, and the initial value for χ is equal to the limit, as $\nu \to \tilde{\nu}$, of either v_+ or v_-, then the arguments in Case (a) apply to show that the new curve is congruent to the original elastic rod.

In the next section we will exhibit concrete examples of new closed constant torsion curves obtained by double Bäcklund transformations of elastic rods of constant torsion.

2.4 Exotic Curves from Elastic Rods

To produce new curves, we need to solve the equations in Prop. (2.5) for the appropriate values of ν. Suppose as before that the rod is of length L and this corresponds to the elliptic parameter x running from zero to $2nK$. Then the formulas in §1.4 give

$$(\mathrm{tr}\Phi(kL))^2 - 4 = \mathrm{tr}(\Psi(kL)^2) - 2 = -4\sin^2(knK\Lambda)$$

when kn is even, and when kn is odd,

$$(\mathrm{tr}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Phi(kL))^2 + 4 = \mathrm{tr}((\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Psi(kL))^2) + 2 = 4\sin^2(knK\Lambda)$$

(Note that while the formulas in §1.4 apply to the linear system (5), the fundamental matrix for (26) differs only by conjugation by matrix A, and so the trace formulas give the same expression.) So, in either case we look for zeros of $\sin(knK\Lambda)$, where Λ is defined by (9) as a function of p and σ, and $\sigma = 2i\nu$ as above.

In practice, we look for values of σ in the first quadrant, because changing ν to $\bar{\nu}$ does not change the double Bäcklund transformation, while changing ν to

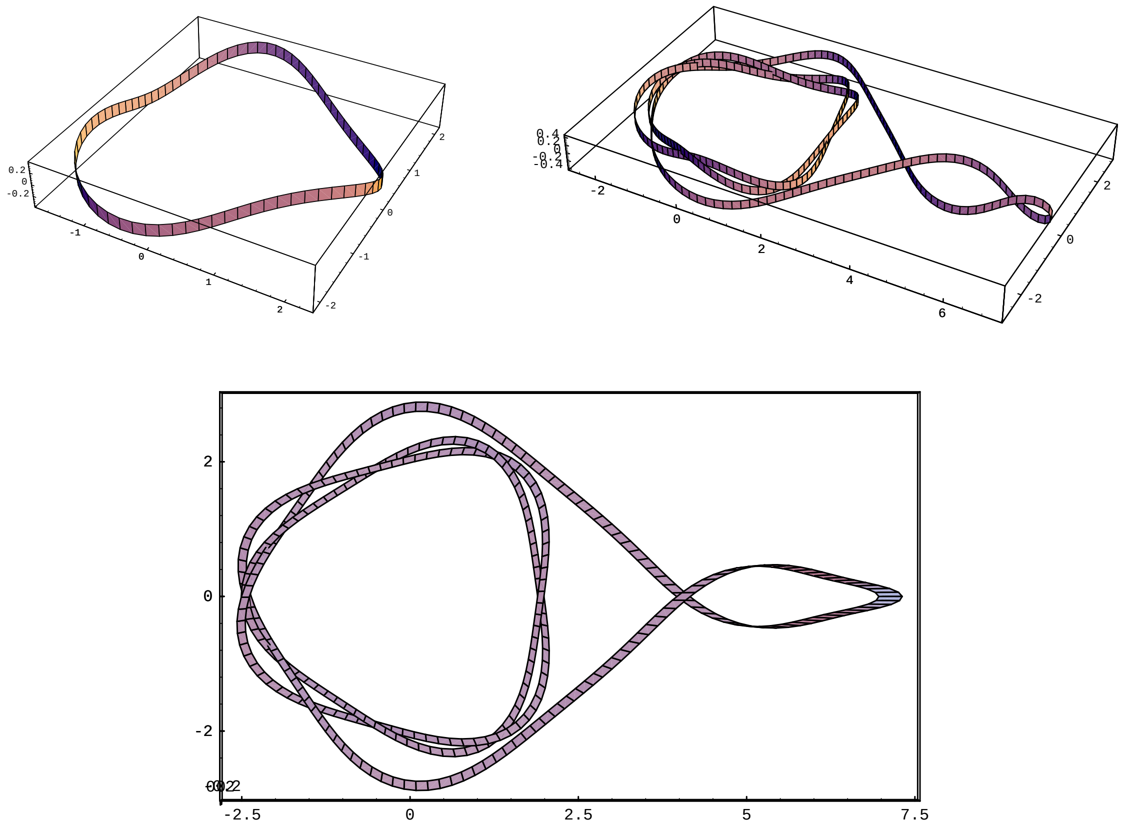

FIGURE 2.1. Knotted double Bäcklund transformation of the unknotted $(1,3)$ elastic rod ($p \approx .63093$, $n = 3$, $k = 4$, $\sigma \approx 1.2283 + 0.9688i$). In the oblique views, the curve itself is the centerline of the vertical ribbon; the elastic rod is at left, the transformed curve at right. In the view from above, the curve itself is the inner edge of the radial ribbon.

$-\nu$ is equivalent, because of symmetries in the linear system, to just changing the sign of $\omega = c_+/c_-$. The formula (10) shows that $\sin^2(K\Lambda)$ has simple zeros at $\sigma = \pm p'/p \pm i$—that is, where $\mu = \sqrt{(p^{-2} - \sigma^2)^2 + 4\sigma^2}/4$ vanishes. (In fact these seem to be the only complex zeros of $\sin^2(K\Lambda)$; there are, however, an infinite number of real zeros, since Λ is asymptotically linear in σ when $\sigma \in \mathbb{R}$ (cf. formula (10)).) We find, using Newton's method, simple zeros for $\sin(knK\Lambda)$ that have real part close to p'/p. Once such a root has been found, we can generate a family of double Bäcklund transformations parametrized by $\omega \in \mathbb{C}$; unless specified, the reader may assume the value used is $\omega = 1$.

We observe the following phenomena:

Number of roots increasing with k: Of course, all the zeros for $\sin(knK\Lambda)$ remain when we multiply k by an integer. However, zeros are also picked up from $\sin(\ell K\Lambda)$ where ℓ is a divisor of kn. For example, when the original curve is the closed unknotted elastic rod of constant torsion corresponding to $p \approx 0.63093$ and $n = 3$ (appearing in Figure 2.1), there is one root for $k = 2$, two new roots for $k = 4$, and four new roots for $k = 8$.

Obtaining knots from unknots: Performing a double Bäcklund transformation of the aforementioned unknot, using one of the roots for $k = 4$, yields the knotted curve of constant torsion shown in Figure 2.1. One can verify that this knot has a minimum crossing number of 12, and is not a torus knot: its Alexander polynomial is

$$A(t) = 1 - t + t^3 - t^4 + t^5 - t^6 + t^7 - t^9 + t^{10},$$

while the only torus knot that has an Alexander polynomial of this degree is a $(2, 11)$-knot, whose polynomial is different. So, this is a new knot type realizable by curves of constant torsion.

Knotting is related to parametric resonance: When, for a fixed initial curve and fixed value of k, there are several roots available, choosing the σ-value closest to $p'/p + i$ produces the Bäcklund transform that has the most complicated shape (see below for examples). At $\sigma = p'/p + i$, the Floquet multipliers coincide but the transfer matrix has only a single eigenvector up to multiple, and the general solution of the linear system is unbounded.

Obtaining unknots from knots: When the original curve is the elastic rod of constant torsion which gives a $(2, 5)$ torus knot, corresponding to $p \approx 0.7845$ and $n = 5$, three different roots are available for $k = 2$. Two of the resulting curves, shown in Figure 2.2, are unknotted.

Self-intersections and dependence on ω: The shape of the transformed curve seems to depend more on the argument of ω than its magnitude. For ω chosen to be real, we sometimes find that the transformed curve has self-intersections. For example, the third available root for the $(2, 5)$ knot with $k = 2$ gives, using $\omega = 1$, the self-intersecting curve with 180-degree symmetry in Figure 2.3. The self-intersection, along with the symmetry, disappears when $\omega = e^{i\pi/3}$. The resulting embedded curve is the twelve-crossing knot shown in Figure 2.3. The shape of the curve changes continuously as the argument of ω varies, showing that a given knot type can have non-congruent realizations as a closed curve of a given constant torsion. Moreover, this curve has the same knot type as the curve in Figure 2.1, showing that this knot type can be realized by closed curves of different constant torsion! (Recall that the Bäcklund transformation preserves the value of the torsion, so these curves have the same torsion as the elastic rods we started with.)

Finally, we note that these exotic curves, produced by double Bäcklund transformations of closed elastic rods of constant torsion, are planar-like 3-solitons—i.e., a linear combination of the purely binormal LIE vector fields B_0, B_1, B_2, B_3 vanishes along them. This follows from our mechanical verification, for $n = 0, 1, 2$, of the conjecture in §1.7, since the calculation is purely algebraic and applies when the Bäcklund parameter C takes successive complex values ν and $\bar{\nu}$.

2.5 Further Research

In this section, we outline some directions for future research.

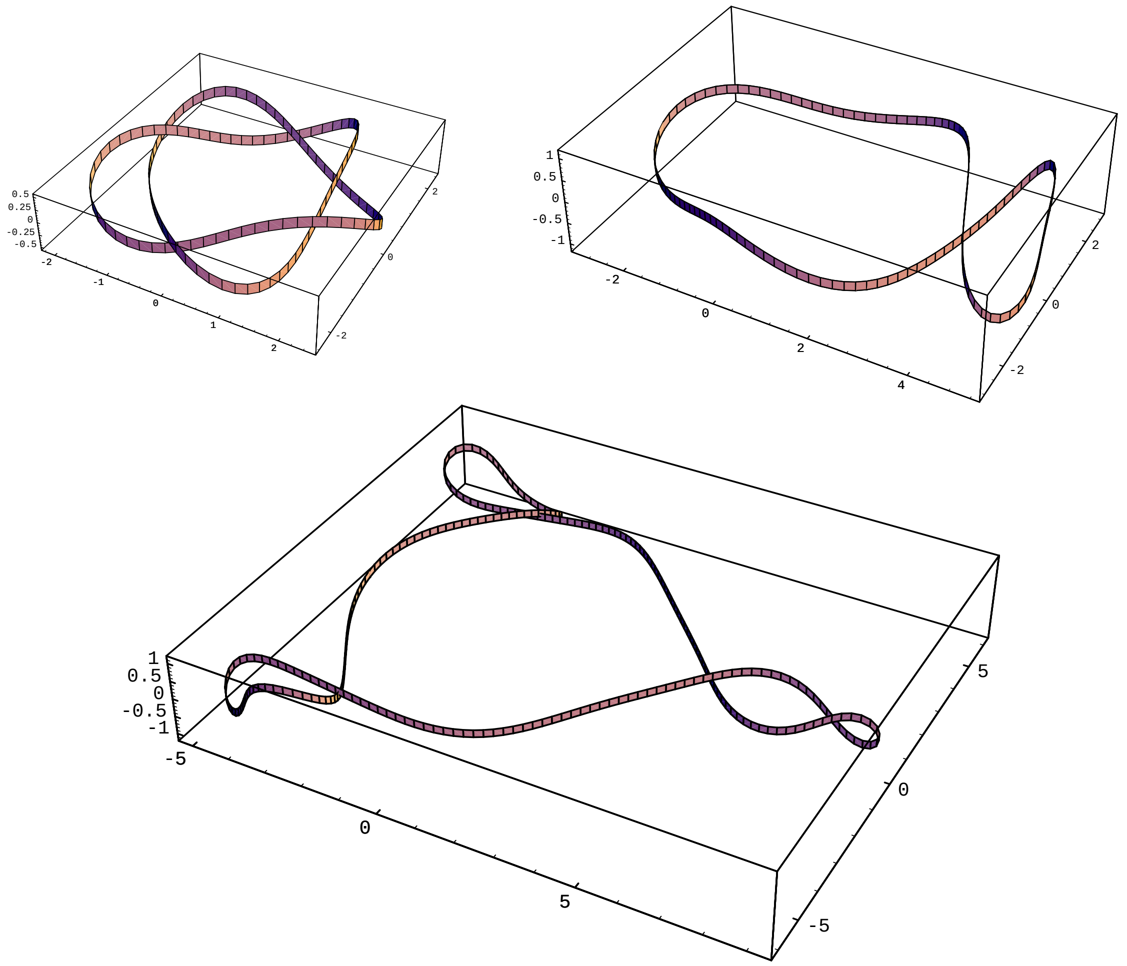

FIGURE 2.2. Unknotted double Bäcklund transformations of the knotted $(2,5)$ elastic rod, with elastic rod shown at above left ($p \approx .7845$, $n = 5$, $k = 2$; $\sigma \approx 0.8982 + 0.8714i$ above right, $\sigma \approx 0.8821 + 0.6716i$ below).

Evolution under the LIE: It would be interesting to see how our exotic curves evolve under the Localized Induction Equation:

$$\frac{\partial \gamma}{\partial t} = \kappa B.$$

Since κB is, up to a tangential component, a Killing field along the elastic rods, they evolve by a rigid motion. (In this case, the LIE preserves constant torsion; we do not expect this to happen in general.) In this context, Ercolani et al [E-F-M] used Bäcklund transformations of a given solution $u(x,t)$ of the sine-Gordon equations to produce solutions that appear as homoclinic orbits, as t goes from $-\infty$ to ∞, in the isospectral set of u. For the LIE itself, the first author [C] has produced homoclinic solutions as Bäcklund transformations of the planar circle (which translates under LIE); moreover, these solutions, which are multiple covers of the circle, exhibit self-intersections that persist under LIE. It is possible that some of

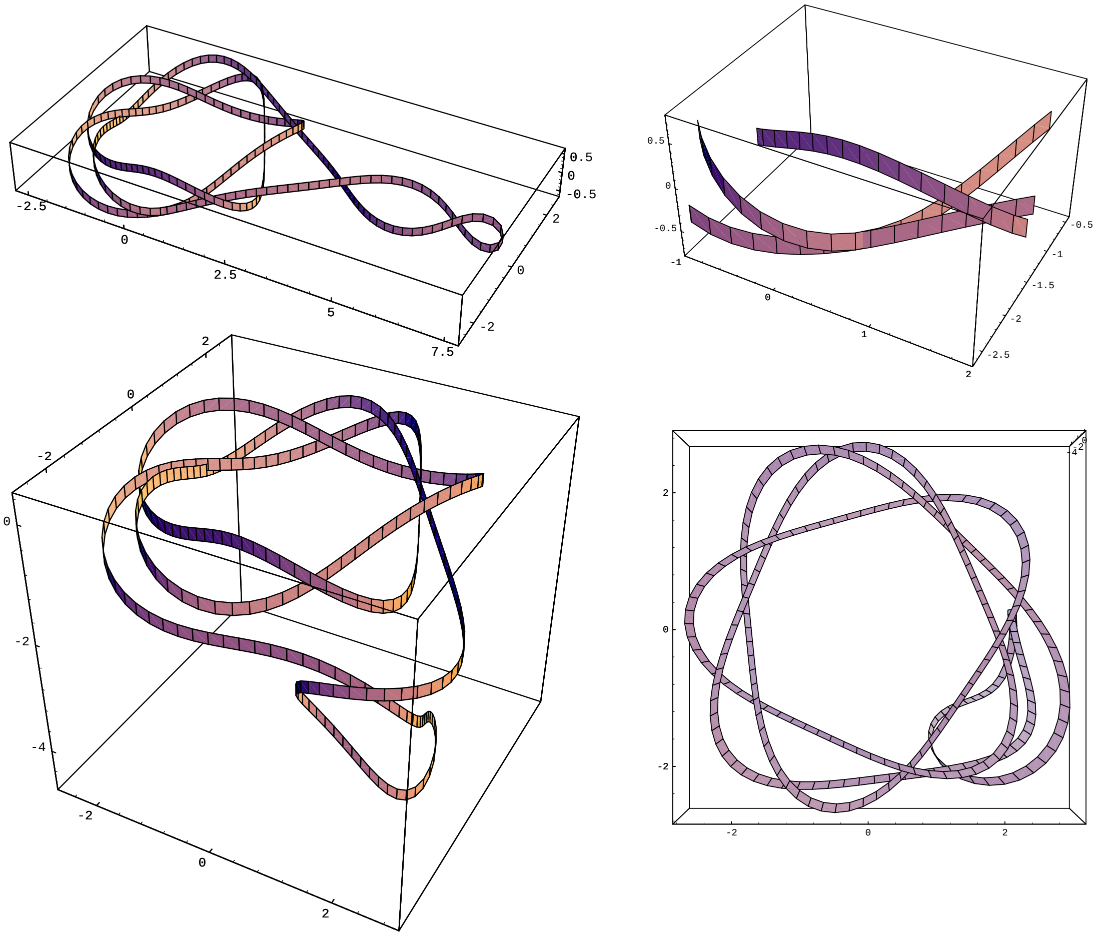

FIGURE 2.3. Resolving self-intersection for a double Bäcklund transformation of the $(2,5)$ elastic rod ($p \approx .7845$, $n = 5$, $k = 2$, $\sigma \approx 0.9067 + 0.9697i$). The upper curve has two self-intersections, one of which is visible in the closeup on the right. The lower curve differs by a change of initial values; at right is a view of it from above.

the self-intersecting LIE solitons produced by our Bäcklund transformations may also exhibit this behaviour.

The space of constant torsion knots: Our examples of exotic curves of constant torsion raise the question of which knot types can be realized with constant torsion. For example, while the trefoil knot is not realized by constant torsion elastic rods (it can, however, be realized by elastic rods of non-constant torsion [I]), is it possible that by a combination of knotting and unknotting, effected by successing double Bäcklund transformations of such elastic rods, a trefoil can be produced? More generally, why, in all of our examples, do all the crossings have the same sign?

Understanding knotting: The examples given in the previous section show that, under the double Bäcklund transformation, the knot type of the curve may change dramatically. One approach to characterizing the change in knot type would be to find invariants which can be computed using the analytic representations we have for these curves. For example, the Möbius energy of the curve, which has been shown to be related to the minimum crossing number [F-H-W], could be computed using the Gauss integral. It may be possible to compute the self-linking number of the new curve and the linking number of the new curve and the old curve. (However, White's formula for the linking number is only valid when the ribbon stretching between the curves is embedded; we were able to use it in §1.6 by taking the Bäcklund parameter $C = \nu$ to be small, but this not true for the roots ν required for the double Bäcklund transformation.) It may also be possible, by generating and examining many more examples, to formulate conjectures concerning the change in knot type, in terms of constructions more familiar to knot theorists (eg. cable knots and satellite knots).

We have shown how the Floquet spectrum of the linear system for an elastic rod contains information about both its knot type (related to the simple roots $\pm\frac{p}{p'} \pm i$ of the Floquet discriminant) and the knot types of its iterated Bäcklund transformations (encoded in some way by the complex double roots). Another approach, then, is to use Floquet analysis in combination with Bäcklund transformations for general multi-phase solutions. Exact formulas for curves of constant torsion that generate n-phase solutions of the LIE or the sine-Gordon equation can be constructed in terms of Theta functions using standard methods of algebraic geometry [Kr, D] which, at the same time, provide a description of the associated Floquet spectrum. (Counting-type of arguments [P-T] can also be used to deduce a priori information about the Floquet eigenvalues.)

Filling up the space of knots: One long-term goal is a complete classification of the knot types of n-phase solutions. In the spirit of Gromov's knot approximation by Legendrian curves [G], it may be possible to combine density theorems for n-phase curves with a precise knowledge of their topology in order to approximate more general knotted curves.

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