RECENT DEVELOPMENTS IN INTEGRABLE CURVE DYNAMICS

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ABSTRACT. The dynamics of vortex filaments has provided for almost a century one of the most interesting connections between differential geometry and integrable equations, and an example in which knotted curves arise as solutions of differential equations possessing remarkably rich geometrical structures. In this paper we present several aspects of the integrable evolution of a closed vortex filament in an Eulerian fluid. Starting with the derivation of the equations from an idealised physical model, we describe their hamiltonian formulation on an appropriate infinite-dimensional phase space. Then we discuss a transformation discovered by Hasimoto which converts the filament equation into the cubic nonlinear Schrödinger equation, unveiling its complete integrability. We work principally with the evolution equation for the tangent indicatrix of the vortex filament, more suitably describing the dynamics of closed curves: this is also a completely integrable soliton equation, originally derived to model the evolution of a continuous Heisenberg spin chain.

This paper has a double task. On the one hand it aims at providing an introduction to soliton equations in a concrete geometrical setting. For this purpose, we discuss an important technique for constructing a large class of special solutions, among which are some interesting torus knots. Also, we make use of the Bäcklund transformation for the Heisenberg spin chain model to explore the symmetries of a given curve and to construct its homoclinic solutions.

On the other hand, some results and techniques of soliton theory are revisited in the light of the differential geometry of curves, where their geometric significance becomes transparent. In this context, we introduce a new orthonormal framing of a given curve and formulate a geometric construction of the Hasimoto transformation. We use this approach to show that the relation between the cubic nonlinear Schrödinger equation and the Heisenberg model is realised by a Poisson map between two different Poisson structures.

1. BACKGROUND

1.1. **The physical model.** Whirlpools and smoke rings are common examples of vortex filaments: approximately one-dimensional regions where the velocity distribution of a fluid has a rotational component. We give below an idealised description of the self-induced dynamics of a closed line vortex in a Eulerian fluid, based on Batchelor's approach [4].

Let $\mathbf{u}$ be the velocity distribution of an incompressible ($\mathrm{div}\mathbf{u} = 0$) fluid filling an unbounded region in space. We suppose that the vorticity $\mathbf{w} = \mathrm{curl}\mathbf{u}$ (measuring the rotational component of the velocity field) is zero at points not on the line vortex. Because $\mathrm{div}\mathbf{u} = 0$, and assuming the infinite domain to be simply connected,

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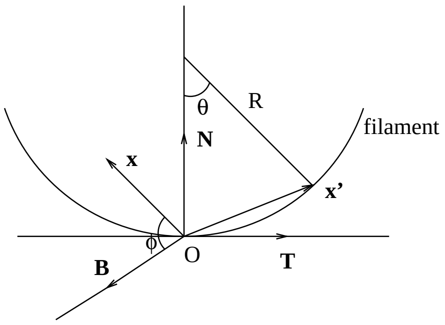


FIGURE 1. Idealised portion of a vortex filament.

we can introduce a vector potential $\mathbf{A}$ and write $\mathbf{u} = \text{curl}\mathbf{A}$. The choice of $\mathbf{A}$ is not unique, in fact $\text{curl}\mathbf{A} = \text{curl}(\mathbf{A} + \text{grad}\phi)$ for any arbitrary scalar function ϕ. We use this freedom to satisfy $\text{div}\mathbf{A} = 0$ and to derive the following vector Poisson equation,

$$\mathbf{w} = \text{curl}(\text{curl}\mathbf{A}) = -\Delta\mathbf{A}, \tag{1}$$

where Δ is the Laplacian operator. In an infinite domain, its solution is

$$\mathbf{A} = \frac{1}{4\pi}\int d^3\mathbf{x}'\frac{\mathbf{w}(\mathbf{x}')}{\|\mathbf{x}-\mathbf{x}'\|},$$

and the corresponding velocity distribution is

$$\mathbf{u} = -\frac{1}{4\pi}\int d^3\mathbf{x}'\frac{(\mathbf{x}-\mathbf{x}')\times\mathbf{w}(\mathbf{x}')}{\|\mathbf{x}-\mathbf{x}'\|^3}. \tag{2}$$

We now make an essential simplification: we model the line vortex with a tube of infinitesimal cross sectional area dA in which the vorticity $\mathbf{w}$ (everywhere tangent to the line vortex) has constant magnitude w. We then have

$$\mathbf{u} = -\frac{1}{4\pi}\int dA\frac{\delta l(\mathbf{x}-\mathbf{x}')\times\mathbf{w}(\mathbf{x})}{\|\mathbf{x}-\mathbf{x}'\|^3} = -\frac{1}{4\pi}\int w dA\oint\frac{(\mathbf{x}-\mathbf{x}')\times\delta\mathbf{l}}{\|\mathbf{x}-\mathbf{x}'\|^3}.$$

Introducing the circulation $\Gamma = \oint\mathbf{u}\cdot\delta\mathbf{l} = \int w dA$, we obtain the expression

$$\mathbf{u} = -\frac{\Gamma}{4\pi}\oint\frac{(\mathbf{x}-\mathbf{x}')\times\delta\mathbf{l}}{\|\mathbf{x}-\mathbf{x}'\|^3}, \tag{3}$$

which describes a velocity distribution that becomes singular along the line vortex. Evidently, the fluid around the line vortex circulates at an increasing speed as it approaches the core of the vortex, causing local cross sections of the vortex tube to rotate about its center without translating. Therefore, we need to compute the asymptotic contribution of the non-circulatory part of the velocity at points infinitesimally close to the vortex filament in order to describe its dynamics.

We approximate a portion of the vortex filament with a circular arc as shown in figure 1., where the Frenet-Serret frame of the curve $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ defines a local coordinate system. We choose $\mathbf{x} = x_2\mathbf{N} + x_3\mathbf{B}$ in the plane normal to the line vortex, $\mathbf{x}' = x_1'\mathbf{T} + x_2'\mathbf{N}$ to be the position vector of a point along the curve, and we let $\sigma = \sqrt{x_2^2 + x_3^2}$ be the distance of $\mathbf{x}$ from the filament. Near the origin O (for small values of the angle θ) we write

$$\mathbf{x}' \simeq l\mathbf{T} + \frac{1}{2}kl^2\mathbf{N}, \qquad \delta\mathbf{l} = d\mathbf{x}' \simeq (\mathbf{T} + kl\mathbf{N})dl, \tag{4}$$

where $l = R\theta$ is the arclength and $k = R^{-1}$ is the curvature of the filament. In this approximation, the integrand in equation (3) becomes

$$\frac{(\mathbf{x}-\mathbf{x}')\times\delta\mathbf{l}(\mathbf{x}')}{\|\mathbf{x}-\mathbf{x}'\|^3} \simeq \frac{-klx_3'\mathbf{T} + x_3\mathbf{N} - (x_2 + \frac{1}{2}kl^2)\mathbf{B}}{[l^2+\sigma^2 - x_2kl^2 + \frac{1}{4}k^2l^4]^{\frac{3}{2}}}dl.$$

This expression is valid at a point $\mathbf{x} = \sigma\cos\phi\mathbf{N} + \sigma\sin\phi\mathbf{B}$ close enough to the vortex filament, and for l varying in a given interval $[-L, L]$; introducing the new variable $m = l/\sigma$ we rewrite equation (3) as

$$\mathbf{u} = \frac{\Gamma}{4\pi}\int_{-\frac{L}{\sigma}}^{\frac{L}{\sigma}} \frac{\sigma^{-1}[\cos\phi\mathbf{B} - \sin\phi\mathbf{N}] + \frac{1}{2}m^2k\mathbf{B}}{\left[1+m^2 - k\sigma m^2\cos\phi + \frac{1}{4}\sigma^2k^2m^4\right]^{\frac{3}{2}}}dl. \tag{5}$$

(We observe that there is no tangential component, the corresponding term in the integrand being an odd function of m.) As $\sigma \to 0$, the denominator of the integrand converges to $(1+m^2)^{\frac{3}{2}}$ and we obtain the following asymptotic expansion

$$\mathbf{u} \simeq \frac{\Gamma}{2\pi\sigma}(\sin\phi\mathbf{B} - \cos\phi\mathbf{N}) + k\frac{\Gamma}{4\pi}\log\frac{L}{\sigma}\mathbf{B}. \tag{6}$$

The $\mathcal{O}(\sigma^{-1})$–term represents the rotational component of the motion which does not produce a displacement of the vortex filament, while the logarithmic term causes the filament to move in the fluid at a large speed in the direction of the binormal. By rescaling the time variable $t \to \dfrac{\Gamma}{4\pi}\log\dfrac{L}{\sigma}t$, and introducing the position vector to the filament $\gamma(x,t)$, we write the following evolution equation

$$\frac{\partial\gamma}{\partial t} = k\mathbf{B}. \tag{7}$$

We observe that the vectorfield is purely local (a point along the filament only "feels" the effect of nearby points) and non-zero only when the filament is curved (there is no self-induced motion of a straight line vortex).

1.2. **The Hamiltonian formulation.** Following a treatment contained in [6], we describe the vortex filament equation (7) derived above as a hamiltonian system on a suitable infinite-dimensional symplectic manifold.

We first need to specify the appropriate phase space. We will restrict consideration to closed curves in space; let then $\mathcal{L}\mathbb{R}^3 = \{\gamma : S^1 \to \mathbb{R}^3\}$ be the space of smooth maps from the circle into $\mathbb{R}^3$ (the *loop space of* $\mathbb{R}^3$). The infinite group $\mathit{Diff}^+(S^1)$ of orientation preserving diffeomorphisms of the circle acts on $\mathcal{L}\mathbb{R}^3$ simply by changing the parametrisation of any given element. For our purposes we wish to identify knots which "look the same" as global objects, however, we want to distinguish identical pairs that have opposite orientations. To this end, we introduce the quotient space $\mathcal{Y} = \mathcal{L}\mathbb{R}^3/\mathit{Diff}^+(S^1)$ as the space of unparametrised oriented loops. $\mathcal{Y}$ is a very singular space, for example it contains curves with an infinite number of self-crossings, or with self-tangencies of infinite order, and at such points the tangent space to $\mathcal{Y}$ cannot be defined. Let $\widehat{\mathcal{X}}$ be the space of loops with finitely many self-crossings and contacts of finite order, then

Theorem 1. [6] *The space of oriented singular knots*

$$\widehat{\mathcal{Y}} = \widehat{\mathcal{X}}/\mathit{Diff}^+(S^1)$$

is a smooth infinite-dimensional manifold modelled on the Frechét space $\mathcal{C}^\infty(S^1, \mathbb{R}^2)$.

The tangent space of $\widehat{\mathcal{Y}}$ can be characterised in the following way: a tangent vector at a point $\boldsymbol{\gamma}$ in $\widehat{\mathcal{Y}}$ is a smooth vectorfield along the curve $\boldsymbol{\gamma}$ that is everywhere normal to $\boldsymbol{\gamma}$ (we quotient out the tangential component which purely reparametrises the curve).

We now define a 2-form on $T\widehat{\mathcal{Y}}$ which endows $\widehat{\mathcal{Y}}$ with a symplectic structure, and allows us to construct hamiltonian flows on it. The following 2-form was introduced by J. Marsden and A. Weinstein in their study of vortex dynamics in Eulerian fluids [42]

$$\omega_{\boldsymbol{\gamma}}(\mathbf{u}, \mathbf{v}) = \int_0^{2\pi} \left(\frac{d\boldsymbol{\gamma}}{dx} \times \mathbf{u} \cdot \mathbf{v} \right) dx, \tag{8}$$

where $\mathbf{u}$, $\mathbf{v}$ are arbitrary tangent vectors at $\boldsymbol{\gamma}$ (i.e. two normal vector fields along $\boldsymbol{\gamma}$) and x is the arclength parameter (we can choose an arbitrary parametrisation of $\boldsymbol{\gamma}$). The integrand in equation (8) is simply the oriented volume of the parallelepiped constructed on the triple $(\boldsymbol{\gamma}, \mathbf{u}, \mathbf{v})$.

It is immediate to verify that ω is a closed 2-form; in fact, we can carry the exterior differentiation inside the integral and observe that the exterior derivative of a volume form on a 3-dimensional manifold is zero. Nondegeneracy also follows easily.

In the case of a finite-dimensional symplectic manifold M (necessarily of even dimension) a symplectic form ω determines a natural isomorphism

$$\begin{array}{ccc} TM & \longrightarrow & T^*M \\ X & \longrightarrow & \omega(X, \cdot) \end{array}$$

between tangent vectors and one-forms (see for example [2]). In particular, ω associates to any smooth function H on M (a hamiltonian) a unique vectorfield X_H (and thus a hamiltonian flow on M) defined by

$$\omega(\cdot, X_H) = dH(\cdot). \tag{9}$$

In an infinite-dimensional setting the correspondence $TM \to T^*M$ may fail to be onto, i.e. not for every hamiltonian functional can we construct an associated hamiltonian vectorfield. This technical point is addressed in [6] for the space $\widehat{\mathcal{Y}}$ and we refer the interested reader to that book for details. Here we simply remark that, among those functionals which admit hamiltonian vectorfields constructed via the correspondence (9) (Brylinski calls such functionals supersmooth) is the *total length* functional

$$\mathcal{L}(\boldsymbol{\gamma}) = \int_0^{2\pi} \left\| \frac{d\boldsymbol{\gamma}}{dx} \right\| dx, \tag{10}$$

defined on arbitrary smooth curves (not necessarily arclength parametrised). The Frechét derivative of $\mathcal{L}(\boldsymbol{\gamma})$, restricted to the representative of $\boldsymbol{\gamma}$ that is arclength parametrised is

$$d\mathcal{L}(\boldsymbol{\gamma}) = -\frac{d^2\boldsymbol{\gamma}}{dx^2}.$$

We check using (9) that the hamiltonian vectorfield for the total length functional $\mathcal{L}$ is given by

$$X_{\mathcal{L}} = \frac{d\gamma}{dx} \times \frac{d^2\gamma}{dx^2}.$$

In fact, using the Marsden-Weinstein symplectic form we compute, for arbitrary $\boldsymbol{\xi}$

$$\omega(\boldsymbol{\xi}, X_{\mathcal{L}}) = \int_0^{2\pi} \frac{d\gamma}{dx} \times \boldsymbol{\xi} \cdot \left(\frac{d\gamma}{dx} \times \frac{d^2\gamma}{dx^2} \right) dx = \int_0^{2\pi} \left(-\frac{d^2\gamma}{dx^2} \cdot \boldsymbol{\xi} \right) dx = d\mathcal{L}[\gamma]\,(\boldsymbol{\xi})\,.$$

Given the hamiltonian vectorfield $X_{\mathcal{L}}$ we write the associated hamiltonian evolution equation

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial x} \times \frac{\partial^2 \gamma}{\partial x^2}, \tag{11}$$

which can be reduced to the vortex filament equation (7) using the Frenet equations for $\boldsymbol{\gamma}$ in the case of nowhere vanishing curvature.

Remark. An immediate consequence of the hamiltonian form (11) is the invariance of the total length during the curve evolution; a stronger result can be proven with a short calculation: local arclength is also preserved during the motion, i.e. $\partial_t \|\gamma_x\| = 0$, therefore the vortex filament moves in time without stretching (an implicit assumption in the study of the physical model). It follows that x and t are independent variables: we will see that the ability to "commute mixed partials" is at the basis of the most interesting properties of the vortex filament equation.

1.3. **Completely integrable infinite-dimensional systems.** This special class of nonlinear partial differential equations gained its importance in the mid-sixties with the coining of the word "soliton" by Kruskal and Zabuski (although "solitonic" behaviour was first observed in 1953 in a seldom acknowledged article by Seeger, Donth and Kochendörfer [51]; we thank an anonymous referee for this historical remark) and with the development of the inverse scattering method for solving the Korteweg-de Vries equation (KdV) by Gardner, Green, Kruskal and Miura.

The several features shared by these nonlinear equations make them the infinite-dimensional analogues of integrable hamiltonian systems in finite dimensions.

In this section we give a brief exposition of the most important properties of integrable PDE's. A good overview of the subject can be found in the book by A. Newell [44] and a very clear exposition is contained in the book by G. L. Lamb [33]; we also refer to M. J. Ablowitz and H. Segur's monograph [1].

Integrable PDE's possess a class of special solutions widely known as solitons. These are solitary waves in the form of pulses whose behaviour has many particle-like features. During their evolution, solitons propagate without change of shape and with no energy loss. When two or more solitons with different propagation speed collide, after a highly nonlinear interaction the pulses emerge with the same initial form and no energy is lost in radiation processes in the course of the interaction.

All the known soliton equations can be considered as infinite-dimensional counterparts of completely integrable finite-dimensional hamiltonian systems in the following sense. They possess an infinite sequence $\{I_k\}_{k=1}^{\infty}$ of constants of motion, whose gradients are linearly independent and whose associated hamiltonian flows commute (i.e. the I_k's are pairwise in involution with respect to a given Poisson bracket on the phase space). For example, the focussing cubic nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0, \tag{12}$$

which will play an important role in the course of this article, can be rewritten as a hamiltonian system

$$\psi_t = \{H, \psi\},$$

with Hamiltonian

$$H[\psi] = \int_{\mathcal{D}} (|\psi_x|^2 - |\psi|^4)\, dx$$

with respect to the following Poisson bracket

$$\{F, G\} = i \int_{\mathcal{D}} \left(\frac{\partial F}{\partial \psi}\frac{\partial G}{\partial \bar{\psi}} - \frac{\partial F}{\partial \bar{\psi}}\frac{\partial G}{\partial \psi} \right)\, dx.$$

The first few invariants are

$$I_1 = \frac{1}{2i}\int_{\mathcal{D}} (\psi_x\bar{\psi} - \bar{\psi}_x\psi)\, dx, \quad I_2 = \int_{\mathcal{D}} |\psi|^2\, dx, \quad I_3 = \int_{\mathcal{D}} \left(|\psi_x|^2 - |\psi|^4\right)\, dx, \dots$$

They satisfy $\{I_k, I_j\} = 0,\ \forall k, j$ and their gradients are linearly independent.

As a consequence of the existence of a hierarchy $\{I_k\}_{k=1}^{\infty}$ of invariants, the phase trajectories are restricted to lie on the intersection of the level sets $I_k = c_k,\ k = 1\dots\infty$, which forms a submanifold of infinite codimension. In the case of a finite-dimensional phase space, the preimage of a regular value $\vec{c}$ of the map $\vec{I} = (I_1, \dots, I_n)$ is diffeomorphic to a product of circles and lines (see [2]) and the dynamical system can be described in terms of the linear evolution of a collection of action-angle variables. For soliton PDE's, the inverse scattering method explicitly constructs a nonlinear change of variables that linearises the flow. The KdV equation was the first soliton equation for which the inverse scattering method was developed: in this case, the analogues of action-angle variables are the scattering data of a related linear Schrödinger operator. The initial value problem for the KdV equation can thus be solved exactly by mapping the initial condition to its scattering data, evolving the scattering data according to a linear evolution up to time t, and by reconstructing the solution at time t of the original PDE by means of the inverse transform.

We mentioned that at the heart of the inverse scattering method for the KdV equation is an underlying linear operator L. Peter Lax [38] proved that the spectrum of the Schrödinger operator L does not vary in in time, and recast the nonlinear PDE into a general framework. If the spectrum of L is independent of time, then its evolution can be written as

$$L(t) = U(t)L(0)U^{-1}(t) \tag{13}$$

for some time-dependent unitary operator $U(t)$. It follows directly from equation (13) that, if some function ϕ solves the eigenvalue problem $L\phi = \lambda\phi$, then ϕ is also a solution of the linear system $\phi_t = B\phi$, where $B = U_tU^{-1}$. Differentiating equation (13) with respect to time gives Lax form of the KdV equation

$$L_t = [B, L],$$

where the operators L and B are called a *Lax pair* for the soliton equation. Lax also showed the existence of an infinite sequence of operators B that give rise to spectrum preserving evolutions of L: the corresponding hierarchy of Lax equations coincide with the infinite sequence of commuting hamiltonian KdV flows mentioned above.

In fact, all completely integrable PDE's arise from a Lax pair, i.e. as solvability conditions of an auxiliary pair of linear systems. The Lax pair for the focussing cubic NLS equation has the following form

$$\begin{aligned} \mathbf{F}_x &= U\mathbf{F} \\ \mathbf{F}_t &= V\mathbf{F}, \end{aligned} \tag{14}$$

where $\mathbf{F}$ is an auxiliary complex vector-valued function and

$$U = i\lambda\sigma_3 + \begin{pmatrix} 0 & i\bar{\psi} \\ i\psi & 0 \end{pmatrix}, V = (2i\lambda^2 - i|\psi|^2)\sigma_3 + \begin{pmatrix} 0 & 2i\lambda\bar{\psi} + \bar{\psi}_x \\ 2i\lambda\psi - \psi_x & 0 \end{pmatrix},$$

with $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (the parameter λ is called the *spectral parameter*).

We conclude this section with a geometric interpretation of the Lax pair (14) given in [20] that is appropriate to this context. The solvability condition for the overdetermined system (14) is obtained by differentiating the first equation with respect to t, the second equation with respect to x and equating the mixed partial derivatives, giving the equation

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0. \tag{15}$$

We can check that equation (15) is identically true for every λ provided the NLS equation (12) is satisfied. We can give to this equation and to the associated linear system (14) a natural geometrical interpretation. The system (14) can be regarded as equations of parallel transport in the trivial vector bundle $\mathbb{R}^2 \times \mathbb{C}^2$, where the vector function $\mathbf{F}$ takes values in the fibre $\mathbb{C}^2$ and the matrices U and V are interpreted as local connection coefficients. Then, equations (14) express the fact that the covariant derivative of $\mathbf{F}$ is zero and equation (15) is equivalent to the (U, V)–connection being a flat connection on $\mathbb{R}^2 \times \mathbb{C}^2$. For this reason, Lax equations of the form (15) are also called the *zero curvature formulation* of the soliton equation.

1.4. The Hasimoto transformation.

In 1972, R. Hasimoto [26] constructed the complex function $\psi = k\exp(i\int^x \tau ds)$ of the curvature k and the torsion τ of a space curve, and showed that if the curve evolves according to the vortex filament equation (7), then ψ solves the focussing cubic nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} + \frac{1}{2}|\psi|^2\psi = 0, \tag{16}$$

which can be reduced to equation (12) of section 1.2 by rescaling ψ to 2ψ. The most direct way to prove this result is to rewrite the filament equation in terms of a new orthonormal frame $(\mathbf{T}, \mathbf{U}, \mathbf{V})$, where

$$\begin{aligned} \mathbf{U} &= \cos(\int^x \tau ds)\mathbf{N} - \sin(\int^x \tau ds)\mathbf{B}, \\ \mathbf{V} &= \sin(\int^x \tau ds)\mathbf{N} + \cos(\int^x \tau ds)\mathbf{B}. \end{aligned}$$

The role of $(\mathbf{T}, \mathbf{U}, \mathbf{V})$ will be the main theme of section 4, for our purposes here we just need the Darboux equations for the new frame

$$\mathbf{T}_x = k_1 \mathbf{U} + k_2 \mathbf{V}, \qquad \mathbf{U}_x = -k_1 \mathbf{T}, \qquad \mathbf{V}_x = -k_2 \mathbf{T}, \tag{17}$$

with $k_1 = k \cos(\int^x \tau ds)$, $k_2 = k \sin(\int^x \tau ds)$. The filament equation (7) can be rewritten in the following form

$$\gamma_t = -k_2 \mathbf{U} + k_1 \mathbf{V}. \tag{18}$$

Differentiating (18) twice with respect to the arclength parameter x, we obtain

$$\gamma_{txx} = -k_{2xx} \mathbf{U} + k_{1xx} \mathbf{V} + (k_1 k_{2x} - k_2 k_{1x}) \mathbf{T}. \tag{19}$$

On the other hand, $\gamma_{xx} = k_1 \mathbf{U} + k_2 \mathbf{V}$ from which we derive

$$\gamma_{xxt} = k_{1t} \mathbf{U} + k_{2t} \mathbf{V} + k_1 \mathbf{U}_t + k_2 \mathbf{V}_t. \tag{20}$$

In order to compare equations (19) and (20), we observe that $\mathbf{T}_t = -k_{2x} \mathbf{U} + k_{1x} \mathbf{V}$ (this follows directly from differentiating (18) with respect to x), and that $\mathbf{U}_t \cdot \mathbf{V} = -\mathbf{U} \cdot \mathbf{V}_t$. Using the Darboux equations for the new frame we compute

$$\begin{aligned} (\mathbf{U}_t \cdot \mathbf{V})_x =& \mathbf{U}_{xt} \cdot \mathbf{V} + \mathbf{U}_t \cdot \mathbf{V}_x = \\ & - k_1 k_{1x} - k_2 \mathbf{U}_t \cdot \mathbf{T} = -k_1 k_{1x} + k_2 \mathbf{U} \cdot \mathbf{T}_t = -k_1 k_{1x} - k_2 k_{2x}. \end{aligned}$$

It follows that $\mathbf{U}_t \cdot \mathbf{V} = -(k_1^2 + k_2^2)/2 + A(t)$, where $A(t)$ is some arbitrary x-independent function. By equating the right-hand sides of (19) and (20), we derive the following pair of equations

$$\begin{aligned} k_{1t} + \left[\frac{1}{2}(k_1^2 + k_2^2) - A(t) \right] k_2 &= -k_{2xx} \\ k_{2t} - \left[\frac{1}{2}(k_1^2 + k_2^2) - A(t) \right] k_1 &= k_{1xx} \end{aligned}$$

which can be written as

$$i\psi_t + \psi_{xx} + \left[\frac{1}{2} |\psi|^2 \psi - A(t) \right] \psi = 0. \tag{21}$$

in terms of the complex function $\psi = k_1 + ik_2$, and can be reduced to the focussing nonlinear Schrödinger equation (16) by changing ψ to $\exp(-i \int^t A(s) ds) \psi$. Therefore, the vortex filament evolution is described by a completely integrable soliton equation. A consequence of this fact is the existence of an infinite hierarchy of global geometric functionals that remain constant throughout the evolution and that are obtained by reexpressing the constants of motion of the NLS equation in terms of the curvature and torsion of the corresponding curve. The first few global invariants of the filament equation are listed below (the first two are not obtained from NLS invariants, as we will see in the next section)

$$I_{-1} = \int_0^{2\pi} \|\gamma_s\| ds, \quad I_0 = \int_0^{2\pi} \tau ds, \quad I_1 = \int_0^{2\pi} \frac{1}{2} k^2 ds,$$

$$I_2 = \int_0^{2\pi} k^2 \tau ds, \quad I_3 = \int_0^{2\pi} \left[(k_x)^2 + k^2 \tau^2 - k^4 \right] ds \ldots.$$

J. Langer and R. Perline [34] unveiled the geometrical significance of the Hasimoto transformation by showing that the NLS equation and the vortex filament equation can be regarded as the same hamiltonian system written with respect to different

Poisson structures. We will explain this interesting result using an intermediate dynamical system introduced in the next section.

1.5. **The Continuous Heisenberg Model.** If we differentiate the vortex filament equation (11) once with respect to the variable x and permute t and x derivatives (the fact that arclength is locally preserved allows us to do that), we obtain the following equation for the unit tangent vector $\mathbf{T}$,

$$\mathbf{T}_t = \mathbf{T} \times \mathbf{T}_{xx}. \tag{22}$$

This equation was derived by M. Lakshmanan [32] and it is known as the Continuous Heisenberg Model (HM). It describes the integrable evolution of the continuum approximation of a discrete spin chain with nearest neighbour interaction (the analogue of the localised potential for the vortex filament dynamics).

The Lax pair for the Heisenberg Model is more conveniently written using matrices rather than spin vectors. If we represent the unit vector $\mathbf{T}^T = (t_1, t_2, t_3)$ with the hermitian trace zero matrix

$$S = \begin{pmatrix} t_3 & t_1 - it_2 \\ t_1 + it_2 & -t_3 \end{pmatrix} \qquad S^2 = I,$$

the equation for the continuous spin chain can then be expressed in the following commutator form

$$S_t = \frac{1}{2i}\left[S, S_{xx}\right], \qquad S(x + 2\pi, t) = S(x, t). \tag{23}$$

It can be easily checked that (23) is the compatibility condition of the following pair of linear systems

$$\begin{aligned} \mathbf{F}_x &= i\lambda S\mathbf{F} \\ \mathbf{F}_t &= (2i\lambda^2 S - \lambda S_x S)\mathbf{F}, \end{aligned} \tag{24}$$

where λ is the complex spectral parameter. The main aspects of the complete integrability of HM, the Hamiltonian formulation, its inverse scattering transform and the hierarchy of conserved functionals are discussed in [20].

Of particular interest to us is its relation with the vortex filament equation and the nonlinear Schrödinger equation. In a short communication J. Langer and R. Perline [35] point out that the HM is to be regarded as the intermediate model between these two evolution equations. In [20] the HM and the NLS equation are shown to be related by a gauge transformation between the respective Lax pairs; since we will provide several geometrical interpretations of this gauge equivalence, we briefly describe it here as a purely algebraic fact and leave its verification to the reader.

Proposition 1.1 (The gauge transformation)**.** *Let V be the unitary matrix which solves the NLS Lax pair (14) at $\lambda = 0$ and which satisfies the additional relation $VSV^{-1} = \sigma_3$. Then, if $\mathbf{F}$ is an eigenvector for the Lax pair (24) of the HM, the vector $V\mathbf{F}$ solves the linear problem for the NLS equation.*

In the course of this article, we will also discuss various properties of the infinite geometry of the HM as they manifest themselves in the context of curve dynamics. We conclude by mentioning two of the advantages of using the HM for modelling closed vortex filaments (besides being itself a model of curve evolution on the 2-sphere). The space of periodic curves is characterised by the following invariant subspace of the HM phase space

$$\mathcal{P}_0 = \left\{ \mathbf{T} : \int_0^{2\pi} \mathbf{T}(x,t)dx = 0, \quad \mathbf{T}(2\pi,t) = \mathbf{T}(0,t) \right\}.$$

On the other hand, establishing which subspace of the NLS phase space corresponds to closed curve is a difficult task: deriving a simple set of conditions on the curvature k and the torsion τ of a space curve for the curve to be closed is still an open problem.

Moreover, the curve $\boldsymbol{\gamma}$ is easily reconstructed as $\boldsymbol{\gamma}(x,t) = \int^x \mathbf{T}(s,t)ds + \mathbf{C}$ (where $\mathbf{C}$ can be chosen to be time-independent), while the reconstruction of the curve from its curvature and torsion (and thus from a solution of the NLS equation) involves solving an inverse problem (the Frenet equations of the curve).

2. Special solutions: methods of algebraic geometry.

Soliton equations, when considered on periodic domains, admit large classes of special solutions which are the analogues of solitary waves for rapidly decreasing initial data on an infinite domain. These solutions are named N-phase solutions since they depend on x and t through a finite number of linear phases. Like the solitary waves, they interact nonlinearly in a particle-like manner and superimpose nonlinearly to produce more general solutions of the boundary value problem.

N-phase solutions of soliton equations such as KdV, NLS and general AKNS systems were first constructed by S. P. Novikov and I. M. Krichever [31]. Their method is a rediscovery of classical Riemann surface techniques developed by J. L. Burchnall and T. W. Chaundy [7] and by H. E. Baker [3] for classifying commutative algebras of scalar differential operators.

A different characterisation of N-phase solutions was given by P. D. Lax (in particularly for the KdV equation) [39]: N-phase solutions are critical points for a linear combination of integrals of motion. This characterisation, although non-constructive except for a very low number of phases, provides an interesting point of view: N-phase solutions lie on low-dimensional sets which topologically are N-tori in function space and which foliate the phase space in a way similar to a finite-dimensional integrable system.

In this section we construct the explicit formulas for N-phase solutions of the HM and the associated curves using methods of algebraic geometry. These curves are interesting from a geometrical point of view: they are critical points of the global geometric invariants $\int_0^{2\pi} \tau ds, \int_0^{2\pi} k^2 ds, \int_0^{2\pi} k^2 \tau ds \ldots$ (among them we find the classical elastic curves). We find the construction of the N-phase solutions of the HM itself interesting, since it gives a clear interpretation of the gauge transformation between the HM and the NLS equation.

2.1. The reconstruction of the curve.

Instead of taking the anti-derivative of the tangent vector, we derive a formula for the position vector of the curve in terms of the eigenfunction of the HM linear problem. In the context of N-phase solutions, this provides one with a straightforward way to obtain the curve. A analogous formula (involving the NLS eigenfunctions) was obtained by A. Sym [54] via a different argument. We start with the following result, whose proof follows from standard ODE theory.

Proposition 2.1. *The solution of the linear problem*

$$\frac{\partial \mathbf{F}}{\partial x} = i\lambda S\mathbf{F} \tag{25}$$

is analytic in $\lambda \in \mathbb{C}$ *for an analytic initial condition* $\mathbf{F}(0;\lambda)$.

Let Φ be the fundamental solution matrix of equation (25). Since it is analytic in the eigenvalue parameter, we can differentiate both sides of the linear system with respect to λ and evaluate them at $\lambda = 0$, obtaining the following formula for S:

$$S = -i\,\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial \lambda}\bigg|_{\lambda=0} \Phi^{-1}|_{\lambda=0}. \tag{26}$$

This expression is a perfect derivative since the columns of the fundamental solution matrix at $\lambda = 0$ are independent of x. Integrating (26) with respect to x we obtain the hermitian matrix

$$\Gamma =: \int^x S(s)ds = -i\,\frac{\partial \Phi}{\partial \lambda}\bigg|_{\lambda=0} \Phi^{-1}|_{\lambda=0}, \tag{27}$$

which represents the position vector of the curve in terms of the eigenfunction matrix Φ of the HM linear problem.

2.2. **The Baker-Akhiezer function.** Multi-phase solutions are associated to a set of data on a Riemann surface. We start with a hyperelliptic Riemann surface Σ of genus g described by the equation

$$y^2 = \prod_{i=1}^{2g+2} (\lambda - \lambda_i). \tag{28}$$

We mark the two points ∞_+, ∞_-, which are permuted by the holomorphic involution $\tau(\lambda, y) = (\lambda, -y)$ exchanging the two sheets. We also choose a set of $g+1$ distinct points $\mathcal{D}_0 = P_1 + \cdots + P_{g+1}$ placed in a generic position (a *non-special divisor*) and not containing $\infty_\pm$.

Let $\lambda(P) : \Sigma \to \mathbb{C} \cup \{\infty\}$ be the hyperelliptic projection to the Riemann sphere; this is a choice of meromorphic function on Σ whose pole divisor is $\infty_+ + \infty_-$ (i.e. whose poles are the preimage of ∞ via the map λ). In neighbourhoods of $\infty_\pm$ we choose the local parameters $k_\pm$ such that

$$(k_\pm)^{-1} = (\lambda(P))^{-1}$$

near $\infty_\pm$ respectively.

The main idea of Krichever is to construct a function $\psi(\lambda)$ on Σ which is uniquely defined by a prescribed behaviour at its singularities and which turns out a posteriori to be the simultaneous solutions of a pair of linear systems. The compatibility condition, i.e. the zero curvature representation of these two linear operators, is a completely integrable non-linear equation for the coefficients. Thus, the construction of such a function provides one both with the non-linear equation, an initial condition and its solution.

Definition 2.1. A Baker-Akhiezer function associated to $(\Sigma, \mathcal{D}_0, \infty_\pm)$ is a function ψ on Σ which:

- is meromorphic everywhere except at $\infty_\pm$ and whose set of poles on $\Sigma\backslash\{\infty_\pm\}$ is $\mathcal{D}_0$,

- has essential singularities at $\infty_\pm$ that locally are of the form $\psi(k_\pm) \sim Ce^{p(k_\pm)}$, where C is a constant and $p(k)$ an arbitrary polynomial with complex coefficients.

For our purposes, we recall the following result [31],

Proposition 2.2. *Suppose that the following technical condition holds:*

Condition 1. The divisor $P_1 + \cdots + P_{g+1} - \infty_+ - \infty_-$ is not linearly equivalent to a positive divisor.

Then, if $p(k) = ikx + iQ(k)t$, with x and t complex parameters with $|x|, |t|$ sufficiently small and $Q(k)$ a given polynomial, the linear vector space of Baker-Akhiezer functions associated to $(\Sigma, \mathcal{D}_0, k_\pm)$ is 2-dimensional and it has a unique basis ψ^1, ψ^2 with the following normalised expansions at $\infty_\pm$:

$$\psi^j(x,t;\lambda) = e^{ik_\pm x + iQ(k_\pm)t} \left(\sum_{n=0}^{\infty} \zeta_n^{j\pm}(x,t) k_\pm^{-n} \right), \quad j = 1,2 \tag{29}$$

where $\zeta_n^{j\pm}$ are functions of the parameters x and t and $\zeta_0^{1+} = 1$, $\zeta_0^{1-} = 0$, $\zeta_0^{2+} = 0$, $\zeta_0^{2-} = 1$.

Remarks. We make two remarks before proving the proposition.

1. For a non-special divisor of degree d (a formal integer linear combination of points on Σ counted with multiplicity) the Riemann-Roch formula (see for example [24]), states that the dimension $h^0(\mathcal{D})$ of the linear space of meromorphic functions on Σ whose pole divisor is $\mathcal{D}$ is

$$h^0(\mathcal{D}) = \begin{cases} 1, & d \leq g, \\ d - g + 1, & d > g. \end{cases} \tag{30}$$

2. Condition 1 can be rephrased as: there exists no non-constant meromorphic function with pole divisor $P_1 + \ldots + P_{g+1}$ which vanishes simultaneously at ∞_+ and at ∞_-.

Proof. Uniqueness: Suppose there are two functions ψ_1 and ψ_2 which satisfy the prescription; then their ratio ψ_1/ψ_2 is a meromorphic function (the essential singularities mutually cancel) whose poles are contained in the zero divisor of ψ_2. The condition of "non-speciality" assures that the dimension of the space of such functions is $2(= g + 1 - g + 1$, according to the Riemann-Roch formula). As regards the normalisation, since $h^0(\mathcal{D}_0 - \infty_+) = h^0(\mathcal{D}_0 - \infty_-) = 1$, we can choose two such functions vanishing respectively at ∞_+ and at ∞_- respectively; because $h^0(\mathcal{D}_0 - \infty_+ - \infty_-) = 0$ (Condition 1), they must be independent.

Existence: We follow an argument given in [46] and exhibit a pair of independent functions on Σ with the correct singularities by constructing them explicitly as ratios of Riemann Theta functions (a nice survey of the features which are relevant in this context can be found in [15]). Let

$$a_1, \ldots, a_g, \quad b_1, \ldots, b_g,$$

be a canonical homology basis for the Riemann surface Σ such that $a_i \cdot a_j = 0, b_i \cdot b_j = 0, a_i \cdot b_j = \delta_{ij}$, and

$$\omega_1, \ldots \omega_g, \qquad \oint_{a_i} \omega_j = \delta_{ij}, \quad i,j = 1, \ldots, g$$

be g normalised holomorphic differentials. We introduce the period matrix B,

$$B_{ij} = \oint_{b_i} \omega_j, \qquad i,j = 1, \dots, g$$

and construct the associated Riemann Theta function

$$\theta(z) = \sum_{n \in \mathcal{Z}^g} \exp i\pi \left(< n, Bn > +2 < n, z >\right), \quad z \in \mathbb{C}^g.$$

The essential behaviour at ∞ is introduced by means of the unique normalised differentials of the second kind η and ζ, which satisfy the following conditions:

1. η and ζ have a single pole at ∞ with local expansions

$$\eta \sim idk, \qquad \zeta \sim idQ(k).$$

(these are dictated by the polynomial dependence of the exponent on the local parameter).

2. (normalisation)

$$\oint_{a_i} \eta = 0, \qquad \oint_{a_i} \zeta = 0.$$

We can now build the following function of $P \in \Sigma$ (where $P_0 \in \Sigma$ is a base point)

$$\tilde{\psi} = \exp\left(x \int_{P_0}^{P} \eta + t \int_{P_0}^{P} \zeta\right) \frac{\theta(\mathcal{A}(P) + Ux + Wt - K - \mathcal{A}(\mathcal{D}))}{\theta(\mathcal{A}(P) - K - \mathcal{A}(\mathcal{D}))}. \tag{31}$$

In this expression $\mathcal{D}$ is a divisor specified below and K is the Riemann constant (see [15]) chosen so that $\theta(\mathcal{A}(P) - K - \mathcal{A}(\mathcal{D}))$ has zeros precisely at $\mathcal{D}$, and

$$\mathcal{A}\left(\sum_{k=1}^{g} Q_k\right) := \sum_{k=1}^{g} \int_{P_0}^{Q_k} \boldsymbol{\omega}$$

is the Abel map ($\boldsymbol{\omega}$ is the vector of holomorphic differentials). The map $\mathcal{A}$ associates to a divisor $\sum_{k=1}^{g} Q_k$ on Σ a point of the complex torus $\mathbb{C}^g/\Lambda$ (the Jacobian of the Riemann surface $Jac(\Sigma)$), where Λ is the $2g$-dimensional lattice spanned by the columns of the matrix $(\mathrm{I}\,|B)$.

The vectors U and W have been introduced to make $\tilde{\psi}$ a well-defined function on Σ. The only indeterminacy is the path of integration which can be modified by adding any integer combination of homology cycles. Taking in account the vanishing condition for η and ζ, this produces the overall factor (the m_k' s are integers)

$$\exp\left(\sum_{k=1}^{g}\left[m_k\left(x\oint_{b_k}\eta + t\oint_{b_k}\zeta\right) - m_k\left(xU_k + tW_k\right)\right]\right).$$

This is equal to 1 if we define the components of the "frequency vectors" U and W to be

$$U_k = \frac{1}{2\pi i}\oint_{b_k}\eta, \qquad \frac{1}{2\pi i}W_k = \oint_{b_k}\zeta.$$

At last we are left to choose the pole divisor $\mathcal{D}$ of degree g and to fix the normalisation. As discussed above, two independent functions are uniquely picked by requiring that one vanishes at ∞_+ and the other at ∞_-; for this purpose we introduce the following

Definition 2.2. Let $\mathcal{D}_\pm$ be the unique positive divisors linearly equivalent to $\mathcal{D}_0 - \infty_\pm$.

The choices $\mathcal{D} = \mathcal{D}_+$ and $\mathcal{D} = \mathcal{D}_-$ in formula (31) produce two independent functions, whose poles are in $\mathcal{D}_+$ and $\mathcal{D}_-$ respectively. In order to make the pole divisor be $\mathcal{D}$ we multiply $\tilde{\psi}^{\pm}$ by a meromorphic function $g_{\pm}(P)$ whose zeros lie in $\mathcal{D}_{\pm} + \infty_{\mp}$ and whose poles lie in the original divisor $\mathcal{D}$. We finally obtain the correct Baker-Akhiezer functions

$$
\begin{aligned}
\psi^{1,2} = &\exp\left[x\left(\int_{P_0}^{P}\eta - \eta_{\pm}^{\infty}\right) + t\left(\int_{P_0}^{P}\zeta - \zeta_{\pm}^{\infty}\right)\right] \times \\
&\frac{\theta(\mathcal{A}(P) + Ux + Wt - K - \mathcal{A}(\mathcal{D}_{\pm}))}{\theta(\mathcal{A}(P) - K - \mathcal{A}(\mathcal{D}_{\pm}))} \times \\
&\frac{\theta(\mathcal{A}(\infty_{\pm}) - K - \mathcal{A}(\mathcal{D}_{\pm}))}{\theta(\mathcal{A}(\infty_{\pm}) + Ux + Wt - K - \mathcal{A}(\mathcal{D}_{\pm}))} \times \frac{g_{\pm}(P)}{g_{\pm}(\infty_{\pm})}.
\end{aligned}
\tag{32}
$$

The constant terms $\eta_{\pm}^{\infty}$ and $\zeta_{\pm}^{\infty}$ in the expansion of the argument of the exponential at $\infty_{\pm}$ have been subtracted to make the leading coefficient of the meromorphic part of the eigenfunction matrix be the identity. $\square$

2.3. **The gauge transformation revisited.** Given the unique basis of the vector space of Baker-Akhiezer functions guaranteed in the previous theorem, we can build unambiguously a function and a corresponding pair of linear operators which will be identified with the Lax pair for the Continuous Heisenberg Model. We have the following result [8]:

Proposition 2.3. *If $Q(k) = 2k^2$ and $\Psi(x,t;P)$ is the matrix of Baker-Akhiezer functions constructed in proposition 2.2*

$$\Psi(x,t;P) = \begin{pmatrix} \psi^1(x,t;P) & \psi^1(x,t;\tau(P)) \\ \psi^2(x,t;P) & \psi^2(x,t;\tau(P)) \end{pmatrix},$$

then the columns of the matrix

$$\Phi(x,t;\lambda) = \Psi(x,t;0)^{-1}\Psi(x,t;\lambda) \tag{33}$$

are linearly independent simultaneous solutions of the following pair of linear systems

$$
\begin{aligned}
\frac{\partial \mathbf{F}}{\partial x} &= i\lambda S\mathbf{F} \\
\frac{\partial \mathbf{F}}{\partial t} &= (2i\lambda^2 S - \lambda S_x S)\mathbf{F},
\end{aligned}
$$

where the matrix S is independent of λ and is given by

$$S(x,t) = \Psi(x,t;0)^{-1}\sigma_3\Psi(x,t;0), \tag{34}$$

with $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The value of Ψ at $\lambda = 0$ defines the normalisation of the eigenfunction Φ at the essential singularity. We have in fact

$$\lim_{\lambda\to\infty}\Phi(x,t;\lambda) = \Psi(x,t;0)^{-1}. \tag{35}$$

Moreover, the choice of $\lambda = 0$ as the point at which the HM Baker-Akhiezer function is normalised is arbitrary, provided it is not chosen to be at the essential singularity.

It is shown in [46] that the matrix $\Psi(x,t;\lambda)$ constructed in proposition 2.2 when $Q(k)=2k^2$ solves the Lax Pair (14) for the focusing nonlinear Schrödinger equation. These considerations lead to the following simple interpretation of the gauge transformation between the HM and the NLS equation:

Corollary 1. *Given the matrix $\Psi(x,t;\lambda)$ of Baker-Akhiezer functions for the NLS Lax Pair (14), the matrix obtained by renormalising Ψ to be the identity matrix at an arbitrary $\lambda_0 \neq \infty$ solves the HM Lax Pair at $\lambda-\lambda_0$. Moreover the potential S can be reconstructed by means of equation (35) with 0 replaced by λ_0.*

The proof of Proposition 2.3 is contained in the following lemma [8],

Lemma 2.1. *Given the Baker-Akhiezer matrix function $\Phi(x,t;\lambda)$ normalised as in Proposition 2.2, there exists a unique pair of matrix differential operators L_1 and L_2 of the following form*

$$L_1=\sum_{\alpha=0}^{1}U_\alpha(x,t)\frac{\partial^\alpha}{\partial x^\alpha},\qquad L_2=\sum_{\alpha=0}^{2}V_\beta(x,t)\frac{\partial^\beta}{\partial x^\beta},\tag{36}$$

such that $\Phi(x,t;\lambda)$ solves simultaneously the linear systems

$$L_1\Phi=\lambda\Phi,\qquad L_2\Phi=\frac{\partial\Phi}{\partial t}.\tag{37}$$

Proof. In order to determine the coefficients of the linear operators L_1 and L_2 we examine the asymptotic behaviour of both columns of Φ at $\infty_\pm$. We define $A(x,t)=:\Psi^{-1}(x,t,0)$ and look at an expansion for Φ in the global coordinate λ of the form

$$\Phi(x,t;\lambda)\sim\left(A(x,t)+\sum_{n=1}^{\infty}\frac{1}{\lambda^n}X_n(x,t)\right)\begin{pmatrix} e^{i(\lambda x+2\lambda^2t)} & 0\\ 0 & e^{-i(\lambda x+2\lambda^2 t)}\end{pmatrix}\tag{38}$$

The operator L_1 is uniquely determined by the requirement

$$(L_1-\lambda\mathrm{I})\Phi(x,t;\lambda)=\mathcal{O}\left(\frac{1}{\lambda}\right)\begin{pmatrix} e^{i(\lambda x+2\lambda^2t)} & 0\\ 0 & e^{-i(\lambda x+2\lambda^2 t)}\end{pmatrix}.\tag{39}$$

In fact, by substituting expression (38) in equation (39) and requiring that the $O(1)$ and $O(\lambda)$ terms are equal to zero, one finds the following recursive system of equations:

$$\begin{aligned} U_1&=-iA\sigma_3A^{-1}\\ U_0&=X_1A^{-1}-U_1A_xA^{-1}-iU_1X_1\sigma_3A^{-1}\end{aligned}\tag{40}$$

which determines U_0 and U_1 as functions of A and X_1. Since $\Phi(x,t;0)=\mathrm{I}$, the multiplicative term U_0 in the expression of L_1 must vanish. We check this in the following way.

We consider the matrix of normalised Baker-Akhiezer functions $\Psi(x,t;\lambda)$ constructed in proposition 2.2 and its asymptotic expansion at ∞,

$$\Psi(x,t;\lambda)\sim\left(\mathrm{I}+\sum_{n=1}^{\infty}\frac{1}{\lambda^n}Z_n(x,t)\right)\begin{pmatrix} e^{i(\lambda x+2\lambda^2t)} & 0\\ 0 & e^{-i(\lambda x+2\lambda^2 t)}\end{pmatrix},$$

Because Φ is obtained by normalising Ψ as in (34), then $Z_n=A^{-1}X_n,\ n=1,\ \dots$. If we substitute the asymptotic expansion of Ψ into an equation of the same form as (39), we can show that Ψ satisfies the following linear eigenvalue problem,

$$-i\sigma_3\frac{\partial\Psi}{\partial x} - \left(\sigma_3 Z_1\sigma_3 - Z_1\right)\Psi = \lambda\Psi, \tag{41}$$

In particular, the coefficient $A^{-1}(x,t) = \Psi(x,t;0)$ satisfies (41) at $\lambda = 0$ which we rewrite as

$$\frac{dA^{-1}}{dx} - i[Z_1,\sigma_3]A^{-1} = 0. \tag{42}$$

We now solve for U_0 in the system (40), use equation (42) and obtain

$$U_0 = -A\left(\frac{dA^{-1}}{dx} - i[Z_1,\sigma_3]A^{-1}\right) = 0. \tag{43}$$

The coefficients of L_2 are determined in the same way. The requirement

$$(L_2 - \frac{\partial}{\partial t})\Phi(x,t;\lambda) = \mathcal{O}(\frac{1}{\lambda})\begin{pmatrix} e^{i(\lambda x+2\lambda^2 t)} & 0 \\ 0 & e^{-i(\lambda x+2\lambda^2 t)}\end{pmatrix}, \tag{44}$$

together with the asymptotic expansion (38), determines the following recursive system of equations for the coefficients V_i:

$$\begin{aligned} V_2 &= -2iA\sigma_3A^{-1} \\ V_1 &= 2X_1A^{-1} - 2V_2A_xA^{-1} - iV_2X_1\sigma_3A^{-1} \\ V_0 &= A_t - V_1(A_x + iX_1\sigma_3) - V_2(A_{xx} + 2iX_{1x}\sigma_3 - X_2). \end{aligned} \tag{45}$$

In order to simplify the expressions for V_1 and V_2 we use the the equation for the time evolution of $\Psi(x,t;\lambda)$, which we can derive by substituting the asymptotic expansion of Ψ in the time linear system (36). We obtain the following λ-independent equation

$$-2i\sigma_3\frac{\partial^2\Psi}{\partial x^2} + 2[Z_1,\sigma_3]\sigma_3\frac{\partial\Psi}{\partial x} + 2i[Z_2,\sigma_3] - 2i[Z_1,\sigma_3]\sigma_3Z_1\sigma_3 - 2\sigma_3Z_{1x}\sigma_3 = \frac{\partial\Psi}{\partial t}.$$

Using the fact that $A^{-1}(x,t) = \Psi(x,t;0)$ satisfies this last equation we substitute the expression for A_t into (45) and obtain

$$\begin{aligned} V_2 &= -2iA\sigma_3A^{-1} \\ V_1 &= -i(A\sigma_3A^{-1})_x \\ V_0 &= 0. \end{aligned}$$

For this choice of U_i's and V_i's the right hand side of the relation

$$(L_j - \lambda \mathrm{I})\Phi(x,t;\lambda) = \mathcal{O}(\frac{1}{\lambda})\begin{pmatrix} e^{i(\lambda x+2\lambda^2 t)} & 0 \\ 0 & e^{-i(\lambda x+2\lambda^2 t)}\end{pmatrix}, \tag{46}$$

gives a pair of Baker-Akhiezer functions whose asymptotic expansions have leading coefficients vanishing at $\infty_\pm$. Because they are unique, they must be identically zero. Therefore $\Phi(x,t,\lambda)$ solves simultaneously the equations

$$(L_j - \lambda \mathrm{I}d)\Phi(x,t,\lambda) = 0, \quad j = 1,2$$

which, by introducing the quantity $S = A\sigma_3A^{-1}$, become the Lax Pair for the Continuous Heisenberg Model. □

2.4. **N-phase curves.** In the previous section, we built a basis of Baker-Akhiezer eigenfunctions for the linear problem of the Heisenberg Model and we expressed the potential S purely in terms of the leading coefficient of the expansion of its meromorphic part at $\lambda = \infty$. We now derive a formula for the corresponding N-phase curves. It is clear from what we have discussed so far that the Baker-Akhiezer function for the Heisenberg Model is specified by $g+1$ poles, two essential singularities over ∞ and its normalisation at 0; it is also clear that normalising at a different point (so far as it does not coincide with one of the poles or the zeros) will affect neither the essential singularity nor the divisor of the meromorphic part. Moreover if the pole divisor $\mathcal{D}$ is in general enough position, Condition 1 can be replaced by the following equivalent

Condition 2. The divisor $P_1 + \ldots + P_{g+1} - 0_+ - 0_-$ is not linearly equivalent to a positive divisor.

Here 0_+ and $0_- = \tau(0_+)$ are the two points on Σ corresponding to $\lambda = 0$ which are exchanged by the hyperelliptic involution. Condition 2 and $\mathcal{D}$ being non-special assure that, among the meromorphic functions with poles in $\mathcal{D}$, there is only one which vanishes at 0_+ and just one which vanishes at 0_- (this assures the existence of two distinct functions with poles in $\mathcal{D}$), but there exists no non-constant function which vanishes at both points (this guarantees the independence of the two and therefore the ability to realise any normalisation at $\lambda = 0$). We introduce the following quantities:

a) $\mathcal{D}^0_+$ (resp. $\mathcal{D}^0_-$), the unique effective divisor which is linearly equivalent to $\mathcal{D} - 0_-$ (resp. $\mathcal{D} - 0_+$).

b) $h_\pm(P)$, a meromorphic function whose divisor is $(h_\pm) = \mathcal{D}^0_\pm + 0_\mp - \mathcal{D}$.

By an argument identical to the one described in Proposition 2.2, we obtain the following result,

Proposition 2.4. *The linear problem associated to the Heisenberg Model is solved by a vector function with the following components*

$$\begin{aligned}
\phi^\pm = {} & \exp\left[x\left(\int_{P_0}^{P}\eta - \eta^0_\pm\right) + t\left(\int_{P_0}^{P}\zeta - \zeta^0_\pm\right)\right]\times \\
& \frac{\theta(\mathcal{A}(P) + Ux + Wt - K - \mathcal{A}(\mathcal{D}^0_\pm))}{\theta(\mathcal{A}(P) - K - \mathcal{A}(\mathcal{D}^0_\pm))}\times \\
& \frac{\theta(\mathcal{A}(0_\pm) - K - \mathcal{A}(\mathcal{D}^0_\pm))}{\theta(\mathcal{A}(0_\pm) + Ux + Wt - K - \mathcal{A}(\mathcal{D}^0_\pm))}\times\frac{h_\pm(P)}{h_\pm(0_\pm)}.
\end{aligned} \tag{47}$$

Moreover, the corresponding eigenfunction matrix

$$\Phi(P) = \begin{pmatrix} \phi^+(P) & \phi^+(\tau(P)) \\ \phi^-(P) & \phi^-(\tau(P)) \end{pmatrix} \tag{48}$$

is normalised to be the identity matrix at $\lambda = 0$.

In formula (47), $(\eta, \zeta, P_0, \boldsymbol{\omega})$ are the same as in proposition 2.2. The terms $\eta^0_\pm = \int_{P_0}^{0_\pm}\eta$ and $\zeta^0_\pm = \int_{P_0}^{0_\pm}\zeta$ have been introduced to produce the correct normalisation at $\lambda = 0$.

Finally, we construct the N-phase curves. We use the reconstruction formula derived in section 3.1 (neglecting the additive constant associated to the normalization of Φ at $x=0$)

$$\Gamma(x,t) = -i\,\frac{\partial \Phi(x,t,\lambda)}{\partial \lambda}\bigg|_{\lambda=0}.$$

The derivatives of the entries of Φ can be expressed in terms of the following functions (and by the sheet information),

$$\frac{\partial \phi^{\pm}}{\partial \lambda}\bigg|_{\lambda=0} = a_{\pm}x + b_{\pm}t + \frac{\partial}{\partial P}\log\theta(\mathcal{A}(P) + Ux + Wt - K - \mathcal{A}(\mathcal{D}^0_{\pm}))\bigg|_{P=0_{\pm}} + c^{\pm},$$

where the $c^{\pm}$'s are constant in x and t depending on the Riemann surface data, $a_{\pm} = \pm\eta(0)$, and $b_{\pm} = \pm\zeta(0)$. We show that both coefficients of the linear terms in x and t are zero. Let λ be real for simplicity (we are just interested in what happens at $\lambda = 0$). In this case the fundamental solution Φ of the spatial linear problem is a unitary matrix. We introduce the *Transfer Matrix*

$$T(x,\lambda) = \Phi(x+2\pi,\lambda)\Phi^{-1}(x,\lambda)$$

which takes a solution evaluated at a given x to its value after one spatial period. It is an easy exercise to show that its trace

$$\Delta(\lambda) = Tr\,[T(x,\lambda)]$$

is invariant both with respect to x and t. If we define the *Floquet eigenfunctions* by means of the following formula

$$\phi^{\pm}(x,\lambda) = e^{i\rho(\lambda)x} f^{\pm}(x,\lambda),$$

where $f^{\pm}(x,\lambda)$ are bounded, periodic functions, one can easily check that the following relation holds between the *Floquet exponent* $\rho(\lambda)$ and $\Delta(\lambda)$,

$$\frac{d\rho}{d\lambda} = \frac{1}{2\pi}\frac{\frac{d\Delta}{d\lambda}}{\sqrt{\Delta^2 - 4}}.$$

By comparing the expression of the Baker-Akhiezer eigenfunction with the one for the Floquet eigenfunction, we can identify the differential η with the derivative of the Floquet exponent,

$$\eta(\lambda) = \frac{d\rho}{d\lambda}(\lambda).$$

As a consequence of the form of the linear system we have the following symmetry

$$\Phi(x,-\lambda) = \Phi(-x,\lambda),$$

from which there follows

$$T(2\pi,-\lambda) = T^{-1}(2\pi,\lambda) = \bar{T}^T(2\pi,\lambda).$$

Since the trace of a unitary matrix is real, we have

$$\Delta(-\lambda) = \Delta(\lambda), \qquad \lambda \in \mathbb{R}.$$

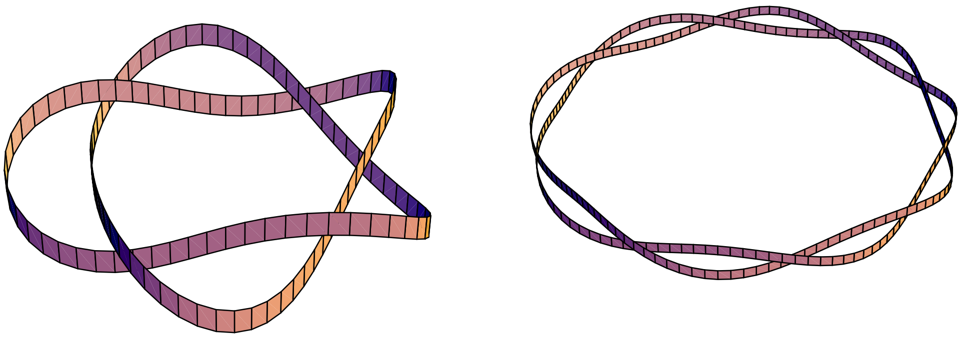
Therefore $\Delta(\lambda)$ is an even function of λ and the Floquet exponent $\rho = \eta$ must vanish at 0. In order to show that the coefficient b vanishes as well, we use the symmetry $\frac{d\phi}{d\lambda}(x,-\lambda) = -\frac{d\phi}{d\lambda}(-x,\lambda)$, where ϕ is any of the components of the eigenmatrix. This formula must hold for all times and in particular in the limit as $t \sim \infty$. At leading order we have $bt = -bt$, it follows that $b = 0$.

Finally we obtain the following compact formula for the components of the hermitian matrix Γ (its left column is the vector $(\Gamma_+, \Gamma_-)^T$ representing the position vector of the curve,

$$\Gamma_\pm = -i \frac{\partial}{\partial P} \log \theta(\mathcal{A}(P) + Ux + Wt + \theta_0)\Big|_{P=0_\pm}, \tag{49}$$

where we absorbed the information about the divisor and the Riemann constant in the initial condition θ_0.

Among the N-phase curves we find planar circles (1-phase solutions that are the analogues of plane wave solutions of the NLS equation) which will play an important role in section 5. Below, we show a pair typical 2-phase curves, constrained extrema of the total squared curvature functional. Solutions to the associated variational problem in the case of unconstrained total torsion (the classical elastic curves) were completely classified by J. Langer and D. Singer [37], and shown to belong to certain classes of torus knots. The curves in figure 2., a 5-2 and a 9-2 torus knot respectively, are critical points of the total squared curvature with constrained total torsion and total length (the most general 2-phase solutions), a vertical ribbon around each of them has been added to make the over and under crossings evident.



(a) (b)

FIGURE 2. A 5-2 torus knot (a) and a 9-2 torus knot (b) are among the 2-phase solutions of the filament equation

3. Natural frames

If the vortex filament equation describes the evolution of an arclength parametrised curve in $\mathbb{R}^3$, the continuous Heisenberg Model describes the corresponding evolution of its unit tangent vector. In the course of this section we will explain how, in order to understand the geometry of the solution space of HM, we need to consider not just the unit tangent vector, but orthonormal frames of the original curve.

The space of unit tangent vectors of space curves is the 2-dimensional sphere of radius 1. The space of associated orthonormal frames can be identified with its unit tangent bundle $\mathcal{T}_1 S^2$. Then, for a given curve in $\mathbb{R}^3$, choosing a orthonormal frame means choosing a way to lift the spherical curve described by the unit tangent vector into the bundle. In this section we study the horizontal lifting associated to the canonical invariant connection on $\mathcal{T}_1 S^2$ and introduce the notion of natural frame. We show that the horizontal lifting is the geometrical realisation of the gauge transformation between the HM and the NLS equation. Moreover, we use this concrete construction to characterise the differential of the gauge transformation in terms of the second symplectic operator for the NLS equation; we will show as a consequence that the HM and the NLS equation can be regarded as the same hamiltonian system written with respect to different symplectic structures belonging to the same hierarchy of hamiltonian flows.

3.1. **The Circle Bundle of $\mathbf{S}^2$.** We start with recalling a few facts about the 2-dimensional sphere and its associated circle bundle. The unit sphere

$$S^2 = \left\{ \mathbf{x} \in \mathbb{R}^3 \ \mid \ x_1^2 + x_2^2 + x_3^2 = 1 \right\}$$

is an example of a *symmetric homogeneous space*. Given a connected Lie group G and a nontrivial group automorphism σ such that $\sigma^2 = \mathrm{Id}$, the associated symmetric homogeneous space is the orbit space G/H, where H is the identity component of the set of elements $h \in G$ which are invariant under the action of σ, i.e. such that $\sigma(h) = h$. In the case of the 2-sphere, G is the Lie group of rotations $SO(3, \mathbb{R})$ and σ is "conjugation by T": $\sigma(g) = T^{-1} g T$, $g \in SO(3, \mathbb{R})$, where T is the matrix $T = \begin{pmatrix} \mathrm{I} & \\ & -1 \end{pmatrix}$. For this choice of the matrix T, H becomes the subgroup of elements of the form $h = \begin{pmatrix} * & \\ & 1 \end{pmatrix}$ which can be identified with the group $SO(2, \mathbb{R})$.

The symmetric homogeneous space $SO(3, \mathbb{R})/SO(2, \mathbb{R})$ is naturally diffeomorphic to the 2-dimensional sphere, we show it by constructing the following transitive action of $SO(3, \mathbb{R})$ on S^2. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis in $\mathbb{R}^3$, the map

$$\begin{array}{ccc} SO(3, \mathbb{R}) & \xrightarrow{\delta} & S^2 \\ g & \longrightarrow & g \cdot e_3 \end{array}$$

induces a diffeomorphism between $SO(3, \mathbb{R})/SO(2, \mathbb{R})$ and S^2 since the subgroup H fixes the vector e_3. Moreover, given the canonical projection $p : SO(3, \mathbb{R}) \to SO(3, \mathbb{R})/SO(2, \mathbb{R}) \cong S^2$, the automorphism σ induces the involution $\sigma_{\mathrm{o}}(px) = p\sigma(x)$, $x \in SO(3, \mathbb{R})$, for which the distinguished point $\mathrm{o} = p\mathrm{e}$ is an isolated fixed point (e denotes the identity element of $SO(3, \mathbb{R})$).

We now define the circle bundle of S^2 as the space of all unit tangent vectors to the sphere:

$$\mathcal{T}_1 S^2 = \left\{ (x, v) \ : \ x \in S^2, \ v \in T_x S^2, \ \|v\| = 1 \right\},$$

($\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^3$) together with the projection $\pi : \mathcal{T}_1 S^2 \to S^2$, $\pi(x, v) = x$ with fibre $\pi^{-1}(x) = S^1$.

Remarks. 1. There is a natural action of the unit circle which "fixes" the base point x and rotates v in the tangent plane to S^2 at x described by the smooth map

$$\begin{array}{ccc} S^1 \times \mathcal{T}_1 S^2 & \longrightarrow & \mathcal{T}_1 S^2 \\ h \cdot (x, v) & \longrightarrow & (x, hv). \end{array}$$

This action is free, i.e. with no fixed points.

2. We can always construct a local cross section, i.e. a smooth map $\phi^{-1} : \pi^{-1}(U) \to U \times S^1$ for some neighbourhood U of each point $x \in S^2$. To do so, we choose a smooth unit vectorfield $e(U)$ (for example, we can take the vectorfield $\partial_u / \|\partial_u\|$ in a local coordinate system (u, v)) and define $\phi(x, h) = (x, he(x))$. Since

the S^1-action is free and $e(x)$ is smooth and never zero in U, ϕ is smooth and invertible and so is ϕ^{-1}.

3. The properties described above define a principal fibre bundle with fibre S^1 and base S^2. The classical result that there does not exist a non-vanishing vector field on S^2 (and therefore a global cross-section) implies that $\mathcal{T}_1 S^2$ is a non-trivial bundle.

4. The transitive, free action of $SO(3,\mathbb{R})$ on $\mathcal{T}_1 S^2$ described by the map $g\cdot(x,v) = (gx, gv)$, $g \in SO(3,\mathbb{R})$, identifies the space of orthonormal frames with the circle bundle of S^2: $SO(3,\mathbb{R}) \cong \mathcal{T}_1 S^2$.

3.2. **The Canonical Connection.** We summarise the notion of a connection in a principal fibre bundle and the construction of the canonical invariant connection for $\mathcal{T}_1 S^2$. The content of this section can be found in various references, ([30] is comprehensive, [52] contains a simple treatment of the circle bundle of a 2-dimensional Riemannian manifold).

As in Euclidean space there is a natural way to parallel-translate and compare vectors at different points, likewise in a general manifold a choice of a connection prescribes a way of translating tangent vectors "parallel to themselves" and to intrinsically define a directional derivative. In the case of a principal bundle P with structure group G over a manifold M

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \big\downarrow \pi \\ & & M \end{array}$$

we best explain the role of a connection when thinking of lifting a vectorfield $v \in TM$ to a vectorfield $\tilde{v} \in TP$ in a unique way. For each $p \in P$, let G_p be the subspace of T_pP consisting of all the vectors tangent to the vertical fibre. The lifting of v will be unique if we require $\tilde{v}(p)$ to lie in a subspace of T_pP complementary to G_p. A smooth and G-invariant choice of such a complementary subspace is called a *connection* on P. More precisely,

Definition 3.1. A connection on a principal bundle P is a smooth assignment of a subspace $H_p \subset T_pP$, $\forall p \in P$ such that:

1. $TP_p = G_p \oplus H_p$
2. $H_{gp} = (\mathcal{L}_g)_* H_p, \qquad \forall g \in G \qquad (\mathcal{L}_g$ is the left-translation in $G)$.

Given a connection, the horizontal subspace H_p is mapped isomorphically by $d\pi$ onto $T_{\pi p}M$. Therefore the lifting of v is the unique horizontal $\tilde{v}$ which projects onto v. An equivalent way of assigning a connection is by means of a Lie algebra valued 1-form ϕ (the connection form). If $A \in \mathfrak{g}$ (the Lie algebra of G), let A^* be the vectorfield on P induced by the action of the 1-parameter subgroup e^{tA}. Since the action of G maps each fibre into itself, then A^* is tangent to the vertical fibre at each point. For each $X \in T_pP$, $\phi(X)$ is the unique $A \in \mathfrak{g}$ such that A^* is equal to the vertical component of X. It follows that $\phi(X) = 0$ if and only if X is horizontal.

Proposition 3.1. *A connection* 1*-form* ϕ *has the following properties:*

1. $\phi(A^*) = A$
2. $(\mathcal{L}_g)^*\phi = Ad_g \phi, \qquad \forall g \in G$ *(Ad is the adjoint representation of G).*

The first property follows immediately from the definition of connection 1-form; for a proof of the second property we refer to [30].

We are now ready to construct an invariant connection on $\mathcal{T}_1 S^2$. The group involution σ defined in the previous section induces a Lie algebra automorphism on $\mathfrak{g} = \mathfrak{so}(3, R)$ (denoted with the same letter σ). The automorphism σ determines the following direct sum decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k},$$

where $\mathfrak{h} = \{X \in \mathfrak{g} | \ \sigma X = X\}$ is the subalgebra of the invariant subgroup $H = SO(2, R)$ and $\mathfrak{k} = \{X \in \mathfrak{g} | \ \sigma X = -X\}$.

Let θ be the canonical 1-form of $SO(3, \mathbb{R})$, i.e. the left-invariant $\mathfrak{g}$-valued 1-form defined by

$$\theta(A) = A, \qquad\qquad A \in \mathfrak{g},$$

then

Theorem 2. *The $\mathfrak{h}$-component ϕ of the canonical 1-form θ of $SO(3, \mathbb{R})$ defines a left-invariant connection on $\mathcal{T}_1 S^2$.*

Proof. Since θ is left-invariant, we restrict to left-invariant vectorfields X on $\mathcal{T}_1 S^2$. Then $\phi(X)$=0 if and only if $X \in \mathfrak{k}$ and $\phi(Y) = Y$ if $Y \in \mathfrak{h}$. θ is left-invariant also with respect to the action of $H = SO(2, R) \cong S^1$; therefore ϕ satisfies both properties 1 and 2 of a connection form and it is invariant under the full action of $SO(3, \mathbb{R})$. □

Structure equations. Let (V, E_1, E_2) be the canonical basis for the Lie algebra $\mathfrak{so}(3, R)$ such that V spans the vertical space $\mathfrak{h}$ and (E_1, E_2) span the horizontal subspace $\mathfrak{k}$. We have

$$[V, E_1] = E_2, \quad [V, E_2] = -E_1, \quad [E_1, E_2] = V.$$

Setting $\theta = \phi V + \omega_1 E_1 + \omega_2 E_2$ and using the Maurer-Cartan equation

$$d\theta(X, Y) = -\frac{1}{2}\theta([X, Y]), \qquad X, Y \in \mathfrak{g},$$

we obtain the following structure equations for the dual basis $(\phi, \omega_1, \omega_2)$

$$d\phi = -\omega_1 \wedge \omega_2, \qquad d\omega_1 = \phi \wedge \omega_2, \qquad d\omega_2 = -\phi \wedge \omega_1.$$

Remarks. 1. If the Riemannian metric on S^2 is inherited from restricting the Euclidean metric on $\mathbb{R}^3$ to S^2, then the invariant connection constructed above coincides with the Riemannian connection on the frame bundle of S^2 (the unique connection which leaves the metric invariant and has zero torsion).

2. We can give an invariant definition of the 1-forms ω_1, ω_2 using the Riemannian structure, the left-invariance of the connection and the isomorphism $d\pi : H_{(x,v)} \to T_x S^2$ between the horizontal subspace at (x, v) and the tangent space to S^2 at x. We identify $T_{\rm o} S^2$ with the horizontal subspace $\mathfrak{h}$. Then (E_1, E_2) may be taken to be the orthonormal basis (e_1, ie_1) of $T_{\rm o} S^2$, where ie_1 is obtained by a 90° rotation of e_1 within the tangent plane. Then for $t \in T_{({\rm o}, e_1)} \mathcal{T}_1 S^2$, $\omega_1|_{({\rm o}, e_1)}(t)$ and $\omega_2|_{({\rm o}, e_1)}(t)$ are the components of the projection $d\pi(t)$ relative to the orthonormal basis (e_1, ie_1), we have for example $\omega_1|_{(o, e_1)}(t) =< d\pi(t), e_1 >$. Since ω_1, ω_2 are invariant with respect to the S^1-action, we can define them on all tangent vectors at o of the form $v = he_1, \ h \in S^1$. The left-invariance of the metric under the full group action defines them everywhere:

$$\omega_1|_{(x,v)}(t) =< d\pi(t), v >, \qquad \omega_2|_{(x,v)}(t) =< d\pi(t), iv >, \qquad t \in T_{(x,v)} \mathcal{T}_1 S^2.$$

3.3. **Horizontal Lifting, Natural Curvatures, Gauge Transformation.** Let $\boldsymbol{\gamma}(s)$ be a smooth curve in $\mathbb{R}^3$ with nonvanishing curvature, $s \in [0, 2\pi]$ be its arclength parameter and $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ be its Frenet frame. The unit tangent vector $\mathbf{T}$ describes a smooth curve $c(s)$ on the unit sphere. In this section we construct the horizontal lifting of $c(s)$ in the bundle $\mathcal{T}_1 S^2$, we introduce the notion of natural curvatures and discuss a geometric interpretation of the gauge transformation between the HM and NLS equations defined in section 1.5.

Given the curve $c(s) = \mathbf{T}(s)$ in S^2, we construct the unique horizontal lifting $\tilde{c}_o = (\mathbf{T}, v)$, with $v(0) = \mathbf{N}(0)$ (i.e. we require that the lifting to the Frenet frame and $\tilde{c}_o$ agree at $s = 0$). In order to compute the vectorfield tangent to $\tilde{c}_o$, we work locally in a coordinate patch $U \subset S^2$ and use the map $\phi \ : \ U \times S^1 \longrightarrow \pi^{-1}(U)$ to identify $\pi^{-1}(U)$ with the product space $U \times S^1$. In a local patch, there exist smooth real-valued functions $\beta(s)$, $\delta(s)$ for which $(\mathbf{T}, \mathbf{N}) \cong (\mathbf{T}, e^{i\beta(s)})$ and

$$\tilde{c}_o(s) \cong (\mathbf{T}(s), h(s)), \qquad \text{with } h(s) = e^{i(\beta(s)+\delta(s))} \in S^1.$$

If ∂_α is the unit tangent vector field to S^1 and $r : \mathbb{R} \to S^1$ is the exponential map $r(\alpha) = e^{i\alpha}$, then $\dfrac{dh}{ds} = \dfrac{d(\beta+\delta)}{ds} dr\left(\dfrac{d}{ds}\right) = \dfrac{d(\beta+\delta)}{ds}\partial_\alpha$. Therefore the velocity field of $\tilde{c}_o$ is

$$\frac{d\tilde{c}_o}{ds} = \left(k\mathbf{N}, \left(\frac{d\beta}{ds} + \frac{d\delta}{ds}\right)\partial_\alpha\right).$$

The lifting $\tilde{c}_o$ is horizontal if and only if the component of its tangent vectorfield along the vertical fibre vanishes, i.e. when $\dfrac{d\beta}{ds} = -\dfrac{d\delta}{ds}$. We will show in the next section (see equation (59))that $\dfrac{d\beta}{ds} = \tau$; using this result and taking count of the initial condition, we require

$$\delta(s) = -\int_0^s \tau(u)du.$$

and we obtain the following expression for the corresponding horizontal lifting

$$\tilde{c}_o(s) = \left(\mathbf{T}(s), e^{-i\int_0^s \tau(u)du}\mathbf{N}\right) = \left(\mathbf{T}, \cos(\int_0^s \tau du)\mathbf{N} - \sin(\int_0^s \tau du)\mathbf{B}\right) \tag{50}$$

(where we have introduced the binormal vector $\mathbf{B} = i\mathbf{N}$). Equation (50) defines the following orthonormal framing of the curve

$$\begin{aligned} \mathbf{U} &= \cos(\int_0^s \tau du)\mathbf{N} - \sin(\int_0^s \tau du)\mathbf{B} \\ \mathbf{V} &= \sin(\int_0^s \tau du)\mathbf{N} + \cos(\int_0^s \tau du)\mathbf{B}. \end{aligned} \tag{51}$$

This is known as the *natural frame* of the curve $\boldsymbol{\gamma}$ (see [5] for some history and discussion); it is indeed "natural" in the sense that it is the lifting of a parallel vectorfield along $\mathbf{T}(s)$ to the orthonormal frame bundle endowed with its canonical connection. The natural frame varies along the curve according to the following system of linear equations

$$\begin{aligned} \frac{d\mathbf{T}}{ds} &= k\cos(\int_0^s \tau du)\mathbf{U} + k\sin(\int_0^s \tau du)\mathbf{V} \\ \frac{d\mathbf{U}}{ds} &= -k\cos(\int_0^s \tau du)\mathbf{T} \\ \frac{d\mathbf{V}}{ds} &= -k\sin(\int_0^s \tau du)\mathbf{T}. \end{aligned} \tag{52}$$

Correspondingly, the components of the projection of the velocity field of the horizontal lifting with respect to $\mathbf{U}$ and $\mathbf{V}$ are called the *natural curvatures*, (we use the invariant expressions of ω_1,ω_2 derived at the end of the previous section)

$$\begin{aligned} k_{\mathrm{U}} &= \omega_1|_{\tilde{c}_o}(\frac{d\tilde{c}_o}{ds}) =< k\mathbf{N}, \cos(\int_0^s \tau du)\mathbf{N} - \sin(\int_0^s \tau du)i\mathbf{N} >= k\cos(\int_0^s \tau du) \\ k_{\mathrm{V}} &= \omega_2|_{\tilde{c}_o}(\frac{d\tilde{c}_o}{ds}) =< k\mathbf{N}, \cos(\int_0^s \tau du)i\mathbf{N} + \sin(\int_0^s \tau du)\mathbf{N} >= k\sin(\int_0^s \tau du). \end{aligned}$$

We can now interpret the gauge transformation that relates HM and NLS in a geometric way: the procedure described above defines a map from curves in S^2 to curves in the complex plane

$$\mathcal{H} : \mathbf{T}(s) \longrightarrow \psi(s) = k(s)\exp{(i\int_0^s \tau du)}.$$

The real and imaginary component of the complex function ψ are the components of the projection of the horizontal vector field onto $T_{\mathbf{T}}S^2$. If $\mathbf{T}$ satisfies the Heisenberg Model equation, then the complex function $\psi(s) = k(s)e^{i\int_0^s \tau du}$ is a solution of the cubic non-linear Schrödinger equation. We summarise this discussion in the

Proposition 3.2. *For frozen t, the gauge transformation between the HM and NLS equations is the composition of the parallel transport of a vector $v(0)$ along the curve $c(s)$ described by $\mathbf{T}$ with its projection onto T_cS^2.*

Remarks. 1. If we parallel transport a different vector (there is a whole circle of choices) the change is reflected in a constant phase factor. The corresponding complex function has the form $\psi = k\exp{[i(\int_0^s \tau du + \beta_0)]}$ for some real constant β_0, which is still a solution of the NLS equation. Thus, choosing a particular initial tangent vector is equivalent to defining a map from the space of smooth curves in S^2 to the space of smooth complex-valued functions quotiented by the action of S^1.

2. A natural frame (a choice of a parallel vectorfield) is always defined, also when the curvature $k(s)$ vanishes, unlike the Frenet frame. In fact, once the vector $v(0)$ is chosen, its horizontal lifting is unique.

3. We also remark that while the Frenet lifting of a closed curve is always closed, the horizontal lifting need not be. Its holonomy is the element of the fibre S^1 which takes the initial value of the lifting to its value at $s = 2\pi$, therefore

$$\mathrm{hol}(\tilde{c}_o) = \exp{(i\oint \tau du)}.$$

Thus, the condition for a closed lifting (i.e. for trivial holonomy) is the following quantisation condition for the total torsion

$$\oint \tau du = 2\pi j, \qquad \text{for } j \in Z.$$

In particular all closed liftings of parallel vectors along a solution of the HM are mapped to periodic solutions of the NLS equation, since $\psi(2\pi) = \psi(0)\exp\left(i\oint \tau du\right)$ while a general lifting is mapped to a quasi-periodic solution. A special case is when the curve γ itself lives on a sphere of radius r, then its total torsion vanishes [12] and its associated natural frame is closed.

3.4. **Comparison between the HM and NLS Phase Spaces.** In this section we compare the phase spaces of the HM and the NLS equation by computing of the differential of the gauge transformation $\mathcal{H}$. Let $\mathcal{L}(S^2) = \left\{c : S^1 \to S^2\right\}$ be the space of smooth closed spherical curves, and let $\Psi = \{\psi : \mathbb{R} \to \mathbb{C}\}$ be the space of smooth complex-valued functions. We introduce the maps

$$\begin{array}{ccc} \mathcal{L}(S^2) & \xrightarrow{\mathcal{H}} & \Psi \\ & & \downarrow \pi \\ & & \Psi/S^1 \end{array}$$

where $\mathcal{H}(\mathbf{T}) = k\exp\left(i\int_0^s \tau du\right)$, and where π is the projection onto the quotient space Ψ/S^1. In this way the composition $h = \pi\circ\mathcal{H}$ maps closed curves in S^2 to complex periodic functions (we factor out the holonomy of the horizontal lifting).

Next we compute the differential of $h = \pi\circ\mathcal{H}$ following a procedure used by J. Langer and R. Perline [34] in deriving a compact expression for the differential of the Hasimoto map. We express dh in terms the second symplectic operator for the NLS equation and relate it to the recursion operator for NLS. As a consequence, we show that the gauge transformation is a Poisson map which carries the Marsden-Weinstein Poisson structure for HM to the second Poisson structure for NLS inducing a corresponding shift in the hierarchy of hamiltonian vectorfields. Firstly, we recall a few facts about Poisson brackets.

Definition 3.2. A Poisson bracket on a manifold M is a bilinear skew-symmetric operation which endows the space of smooth functions on M with a Lie algebra structure; i.e. it is a bilinear map defined on smooth functions on M

$$\{\ ,\ \}:\ \mathcal{C}^\infty(M)\times\mathcal{C}^\infty(M),\longrightarrow M$$

satisfying the following properties:

$$\begin{aligned} &\{f,g\} = -\{g,f\} && \text{(Skew symmetry)}\\ &\{f,gh\} = g\{f,h\} + \{f,g\}h && \text{(Leibnitz rule)}\\ &\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0 && \text{(Jacobi Identity)} \end{aligned}$$

Remarks. 1. If M is a Riemannian manifold (i.e. if TM is endowed with an inner product $<\ ,\ >_m$) then any skew-symmetric linear operator $J(m)$ on tangent vectorfields induces a skew-symmetric bilinear map (not necessarily a Poisson bracket) of the form

$$\{f,g\}(m) = < J\nabla f, \nabla g > (m), \qquad m\in M. \tag{53}$$

(in this case the Leibnitz rule is automatically satisfied).

2. If equation (53) defines a non-degenerate Poisson bracket, then the 2-form $< J\ ,\ >$ is a symplectic form on M (the Jacobi identity is equivalent to such form being closed) and the operator J is called a symplectic operator. For example, the symplectic structure introduced on $\mathcal{L}(S^2)$ in section 1.2 is obtained in this way by

using the inner product

$$< X, Y >_{\mathcal{L}(S^2)} (\mathbf{T}) = \frac{1}{2\pi} \int_0^{2\pi} (X \cdot Y)(s) ds$$

and the skew-symmetric operator

$$J|_{\mathbf{T}} X = \mathbf{T} \times X \qquad \mathbf{T} \in S^2, \quad X \in T_{\mathbf{T}}\mathcal{L}(S^2).$$

The resulting Poisson structure is known as the Marsden-Weistein Poisson bracket

$$\{f, g\}_{\mathrm{MW}}(\mathbf{T}) = \frac{1}{2\pi} \int_0^{2\pi} \nabla g \cdot (\mathbf{T} \times \nabla f)(s) ds.$$

If $f : M \to N$ is a differentiable map, we can obtain an explicit formula for its differential that is suitable for computations, by using the notion of directional derivative. Let $g : N \to \mathbb{R}$ be an arbitrary differentiable function on N, then the action of the vector field V on the real-valued function $g \circ f$ is described by $V_m(g \circ f) =: V[g \circ f](m)$, where the right hand side is the directional derivative of $g \circ f$ along V at the point m. Thus, we write the following expression for $df : TM \to TN$:

$$df(V)[g] = V(g \circ f).$$

Before computing the differential of $\mathcal{H}$ we need the following variational formulas:

Lemma 3.1. *Consider a family of arclength parametrised curves $\gamma(w, s) : (-\epsilon, \epsilon) \times S^1 \to \mathbb{R}^3$ and let k and τ be the curvature and torsion of $\gamma(0, s)$. Let $\mathbf{T}(w, s)$ be the family of spherical curves described by the the unit tangent vector of γ and denote with $W = \mathbf{T}_w|_{(0,s)}$ the variation vectorfield along $\mathbf{T}$. Then, the variation of k and τ along W are given by*

$$W(k) = < W_s, \mathbf{N} > \tag{54}$$

$$W(\tau) = < \nabla_{\mathbf{N}} W, \mathbf{B} >_s + k < W, \mathbf{B} > . \tag{55}$$

In formulas (54) and (55) ∇ is the symbol of covariant differentiation and the subscript s denotes the derivative with respect to s (i.e. the covariant derivative $\nabla_{k\mathbf{N}}$ along the velocity vector of $\mathbf{T}$).

Proof. Since the velocity field $\mathbf{T}_s|_{w=0} = k\mathbf{N}$ of $\mathbf{T}$ commutes with the variation W, we have the sequence following identities

$$0 = [W, V] = \nabla_W (k\mathbf{N}) - \nabla_{k\mathbf{N}} W = W(k)\mathbf{N} + k\nabla_W \mathbf{N} - k\nabla_{\mathbf{N}} W. \tag{56}$$

Given $k^2 =< k\mathbf{N}, k\mathbf{N} >$, we take the covariant derivative of both sides of this equation in the direction of W and use (56) to obtain

$$2kW(k) = 2 < \nabla_W (k\mathbf{N}), k\mathbf{N} >= 2k < \nabla_{k\mathbf{N}} W, \mathbf{N} >= 2k^2 < \nabla_{\mathbf{N}} W, \mathbf{N} > .$$

Formula (54) follows directly together with the following useful relations,

$$[W, \mathbf{N}] = - < \nabla_{\mathbf{N}} W, \mathbf{N} > \mathbf{N} \tag{57}$$

$$\nabla_W \mathbf{N} = \nabla_{\mathbf{N}} W - < \nabla_{\mathbf{N}} W, \mathbf{N} > \mathbf{N} =< \nabla_{\mathbf{N}} W, \mathbf{B} > \mathbf{B}. \tag{58}$$

As for $W(\tau)$, we write $\tau^2 =< \tau\mathbf{B}, \tau\mathbf{B} >$, where $\tau\mathbf{B} = \nabla_{k\mathbf{N}} \mathbf{N}$. Using the chain rule and formula (56) we compute

$$\begin{aligned} 2\tau W(\tau) =& 2\tau < \nabla_W (\nabla_{k\mathbf{N}} \mathbf{N}), \mathbf{B} >= \\ & 2\tau < \left(\nabla_{k\mathbf{N}} \nabla_W + \nabla_{[W, k\mathbf{N}]} + R(W, k\mathbf{N})\right) \mathbf{N}, \mathbf{B} > . \end{aligned}$$

We have used equation

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R(X,Y)Z,$$

where $R(X,Y)$ is the Riemann curvature tensor. For the unit sphere we have $R(X,Y)Z =< Z, Y > X- < Z, X > Y$, there follows

$$\begin{aligned} W(\tau) = &< \nabla_{k\mathbf{N}} \nabla_W \mathbf{N} + kW - < \mathbf{N}, W > k\mathbf{N}, \mathbf{B} >= \\ &< \nabla_{k\mathbf{N}} \left(< \nabla_{\mathbf{N}} W, \mathbf{B} > \mathbf{B}\right), \mathbf{B} > +k < W, \mathbf{B} >= \\ &k\mathbf{N}(< \nabla_{\mathbf{N}} W, \mathbf{B} >) + k < W, \mathbf{B} >= \\ &< \nabla_{\mathbf{N}} W, \mathbf{B} >_s +k < W, \mathbf{B} > . \end{aligned}$$

□

Remark. In the set-up of the previous section $\mathbf{N}$ belongs to the vertical fibre over $\mathbf{T}$ in $\mathcal{T}_1 S^2$. Then, in a local representation $\mathbf{N}$ is identified with some element $e^{i\beta}$ of S^1, where β is a smooth real-valued function. Thus, the action of the vector field W on $\mathbf{N}$ can be written in terms of the unit tangent vector along the vertical fibre ∂_α as $\nabla_W \mathbf{N} = W(\beta)\partial_\alpha$. Comparing this expression with formula (58) and identifying $\mathbf{B}$ with ∂_α ($\mathbf{B}$ is a unit vector tangent to the fibre at $\mathbf{N}$), we compute $W(\beta) =< \nabla_{\mathbf{N}} W, \mathbf{B} >$. In particular, if $W = k\mathbf{N}$, the velocity field of the curve, we obtain

$$\nabla_{k\mathbf{N}}(\beta) = \beta_s = \tau. \tag{59}$$

Next we compute the differential of the map $h = \pi \circ \mathcal{H}$ and reexpress it in terms of the second Poisson structure for NLS. We first recall the following result which can be found in [40] and [20]:

Theorem 3. *There exist two compatible symplectic operators for the periodic NLS equation:*

$$\tilde{J}\phi \;=\; i\phi \tag{60}$$

$$\tilde{K}_\psi \phi \;=\; \phi_s + \frac{1}{4}\psi \left(\int_0^s + \int_{2\pi}^s \right) [\phi\bar{\psi} - \bar{\phi}\psi] du, \tag{61}$$

with respect to the inner product

$$< \phi_1, \phi_2 >|_\psi = \frac{1}{2\pi} \int_0^{2\pi} (\phi_1 \bar{\phi_2} + \bar{\phi_1}\phi_2)\, du \qquad \phi_1, \phi_2 \in T_\psi \Psi. \tag{62}$$

The associated recursion operator $\tilde{\mathcal{R}} = \tilde{K}_\psi \tilde{J}^{-1}$ generates the following infinite hierarchy of Poisson structures

$$\{f, g\}_n =< \tilde{\mathcal{R}}^n \tilde{J} \nabla f, \nabla g >, \quad f, g \in \Psi \tag{63}$$

and the sequence of Hamiltonian vectorfields

$$X_n = \tilde{\mathcal{R}}^n X_0, \qquad X_0 = \psi_x$$

Remark. It is easy to check that the NLS equation can be written as a hamiltonian system with respect to two different hamiltonians using the operators $\tilde{J}$ and $\tilde{K}$. Whenever this happens and the two operators are compatible ($\tilde{J}$ and $\tilde{K}$ are compatible if $\tilde{J} + \tilde{K}$ defines a Poisson bracket) we say that $\tilde{J}$ and $\tilde{K}$ define a bihamiltonian structure on Ψ. A bihamiltonian formulation appears to be one of the unifying properties of soliton equations. Given a bihamiltonian structure, one can formally construct an infinite sequence of constant of motions; it is still an open question though, whether every completely integrable equation admits a bihamiltonian structure (this very question does not yet have a rigorous answer for the periodic HM itself!). The reader who is interested in this aspect of integrability is referred to [45, 40, 10].

Finally we compute the differential of $\mathcal{H}(\mathbf{T}) = k \exp\left(i \int_0^s \tau du\right)$. Using the chain rule and lemma 3.1 we write

$$\begin{aligned} d\mathcal{H}(W) =& W(k) e^{i\int_0^s \tau du} + ik \left(\int_0^s W(\tau) du \right) e^{i \int_0^s \tau du} = \\ & (< W_s, \mathbf{N} > +i < W_s, \mathbf{B} >) e^{i \int_0^s \tau du} + i\psi \int_0^s < W, k\mathbf{B} > du + ic\psi, \end{aligned}$$

c is a constant of integration, and since the kernel of $d\pi$ contains vectorfields of the form $c\psi$, we can write equivalently

$$d\mathcal{H}(W) = (< W_s, \mathbf{N} > +i < W_s, \mathbf{B} >) e^{i \int_0^s \tau du} + \frac{i}{2}\psi \left(\int_0^s + \int_{2\pi}^s \right) < W, k\mathbf{B} > du.$$

We rewrite $W = g\mathbf{U} + h\mathbf{V}$ on the basis of the natural frames and introduce the complex vector field $\eta(W) = g + ih$. Using the Darboux equations (52) for the components of the natural frame, we can rewrite the above expression as

$$d\mathcal{H}(W) = \frac{d}{ds}\eta(W) + \frac{1}{4}\psi \left(\int_0^s + \int_{2\pi}^s \right) [\eta(W)\bar{\psi} - \bar{\eta}(W)\psi] du. \tag{64}$$

The right-hand side of (64) is nothing but the second Poisson operator K_ψ for the NLS equation acting on the tangent vector $\eta(W)$; we summarise this in the following

Proposition 3.3. *The differential of the composed map $h : \mathcal{L}(S^2) \to \Psi/S^1$, $h = \pi \circ \mathcal{H}$ is given by:*

$$dh = d\pi \circ \tilde{K}_{h(\mathbf{T})} \circ \eta. \tag{65}$$

An important consequence of formula (65) is that h preserves Poisson structures, in particular it carries the Marsden-Weinstein Poisson bracket into the second Poisson bracket for the NLS equation. In order to show this we need the following technical result:

Lemma 3.2.

$$\begin{aligned} \nabla(f \circ h)|_{\mathbf{T}} &= dh^*(\nabla f)|_{h(\mathbf{T})}, \qquad f \in \Psi/S^1 \\ \eta \circ J \circ \eta^* &= \tilde{J}. \end{aligned}$$

Here η^* is the adjoint of η with respect to the inner product (62), J is the symplectic operator $\mathbf{T}\times$ of HM and $\tilde{J}$ is simply multiplication by i. The proof of this lemma is a one line computation (in a similar context see [34]) and it is left to the reader.

Corollary 2. *The map h is a Poisson map and the following relation holds*

$$\{f \circ h, g \circ h\}_{MW}(\mathbf{T}) = \{f, g\}_2(h(\mathbf{T})), \tag{66}$$

for $f, g \in \Psi/S^1$.

Proof.

$$
\begin{aligned}
\{f\circ h, g\circ h\}_{\mathrm{MW}}(\mathbf{T}) = &< J\nabla(f\circ h), \nabla(g\circ h) > (\mathbf{T}) = \\
&< J\circ dh^*\nabla f, dh^*\nabla g > (h(\mathbf{T})) = \\
&< dh\circ J\circ dh^*\nabla f, \nabla g > (h(\mathbf{T})) = \\
&< d\pi\circ \tilde{K}\circ \eta\circ J\circ \eta^*\circ \tilde{K}^*\circ d\pi^*\nabla f, \nabla g > (h(\mathbf{T})) = \\
&< \tilde{K}\circ \tilde{J}\circ \tilde{K}^* d\pi^*\nabla f, d\pi^*\nabla g > (h(\mathbf{T})) = \\
&< \tilde{R}^2\tilde{J}\nabla(f\circ\pi), \nabla(g\circ\pi) > (\mathcal{H}(\mathbf{T})) = \\
&\{f\circ\pi, g\circ\pi\}_2(\mathcal{H}(\mathbf{T})) = \{f,g\}_2(h(\mathbf{T})).
\end{aligned}
$$

□

(We used the skew-symmetry of $\tilde{K}$ and the identity $\tilde{J}^{-1} = \tilde{J}$.) Therefore, we can view the NLS equation and the HM as the same hamiltonian system written with respect to two different Poisson structures which belong to the same integrable hierarchy.

4. Bäcklund transformation and immersed knots

A fundamental algebraic property of soliton equations is the fact that they arise as compatibility conditions for a pair of linear operators. In section 3.2 we discovered a deep relation among the infinite family of multi-phase solutions, the nonlinear equation and its associated Lax pair in the case of the NLS and HM equations. The zero-curvature formulation of integrable partial differential equations is at the basis of the inverse scattering method for finding exact solutions (see for example [20]). Moreover, the hierarchy of conserved quantities and the related conservation laws can be constructed in a purely algebraic way through the knowledge of the Lax pair, independently of global properties such as the boundary conditions and the differentiability class of the solutions. Another seemingly universal feature of soliton equations, also intimately connected to the presence of a Lax pair, is the existence of Bäcklund transformations. A Bäcklund formula produces more complicated solutions of the nonlinear equation from a simpler solution for which the pair of linear systems can be solved exactly. The close relation among inverse scattering, the infinite number of conservation laws and Bäcklund transformations was first discussed in 1974 in an elementary and intriguing paper by M. Wadati, H. Sanuki and K. Konno [55]. In their 1983 papers [21, 22, 23], H. Flaschka, A. Newell and T. Ratiu developed a general algebraic framework in which to interpret the unifying properties of soliton equations. The central idea is to view the hierarchy of integrable equations as a sequence of commuting flows on an infinite-dimensional loop algebra (a Kac-Moody Lie algebra, see the original papers or Chapter 5 in [44]), and to think of the solution as a function of an infinite number of independent variables, none of which playing a distinguished role (x is now just one of the time variables).

In this section, we will simply focus on the use of Bäcklund transformations to construct new interesting solutions of soliton equations from simpler solutions. It is instructive to start with describing the Bäcklund formula for the nonlinear Schrödinger equation

$$iq_t = q_{xx} + 2|q|^2 q, \tag{67}$$

to be considered together with its associated Lax pair

$$
\begin{aligned}
&\frac{\partial}{\partial x}\mathbf{F}_{\mathrm{NS}} = \left[i\lambda\sigma_3 + i\begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix}\right]\mathbf{F}_{\mathrm{NS}} \\
&\frac{\partial}{\partial t}\mathbf{F}_{\mathrm{NS}} = \left[(2i\lambda^2 - i|q|^2)\sigma_3 + \begin{pmatrix} 0 & 2i\lambda\bar{q} + \bar{q}_x \\ 2i\lambda q - q_x & 0 \end{pmatrix}\right]\mathbf{F}_{\mathrm{NS}}.
\end{aligned} \tag{68}
$$

The Bäcklund transformation for equation (67) is obtained in the following way (the proof is a direct verification, see [50] for its derivation in the context of gauge transformations). Let $(\boldsymbol{\psi}_+, \boldsymbol{\psi}_-)$ be two independent solutions of the linear system (68) at (q, ν). We construct the following quantities (where $c_\pm$ are complex arbitrary constants)

$$
\begin{aligned}
\boldsymbol{\psi} &= c_+\boldsymbol{\psi}_+ + c_-\boldsymbol{\psi}_- \\
N_{\mathrm{NS}} &= \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} \\
G_{\mathrm{NS}} &= N_{\mathrm{NS}}\begin{pmatrix} \lambda - \nu & 0 \\ 0 & \lambda - \bar{\nu} \end{pmatrix} {N_{\mathrm{NS}}}^{-1}.
\end{aligned}
$$

Then

$$
\mathbf{F}^{(1)}_{\mathrm{NS}}(x, t; \lambda, \nu) = G_{\mathrm{NS}}(\lambda, \nu, \boldsymbol{\psi})\mathbf{F}_{\mathrm{NS}}(x, t; \lambda) \tag{69}
$$

solves equation (68) at $(q^{(1)}, \lambda)$, where

$$
q^{(1)} = q + 2(\nu - \bar{\nu})\frac{\psi_1\bar{\psi}_2}{|\psi_1|^2 + |\psi_2|^2} \tag{70}
$$

is the corresponding new solution of the NLS equation.

Bäcklund transformations can be used to generate homoclinic orbits of a given solution. The importance of understanding the homoclinic manifolds of N-phase solutions that possess nonlinear instabilities is discussed in the work of N. Ercolani, G. Forest and D. McLaughlin [16, 17, 18]. Unstable N-phase solutions, just like saddle points and periodic orbits of saddle-type in finite-dimensional integrable systems, reside on singular level sets in phase space. A complete topological description of the singularities of the foliation of the space of N-phase solutions can be achieved by means of Bäcklund transformations. Bäcklund formulas produce explicitly the homoclinic manifolds and thus label the underlying level sets. Moreover, using Bäcklund transformations one is able to construct both the tangent and normal vectorfields to the level set of a given solution and to generate families of solutions residing on the same level set. In fact [17], a finite number of iterated Bäcklund transformations generates the entire level set of a given N-phase solution.

These ideas are hereby applied to curves in space, a context in which the relevant structures can be more easily visualised. We will use Bäcklund formulas in a twofold way. Firstly, we will construct curves which are homoclinic to a given curve and conjecture that such homoclinic curves play a role in separating different classes of knots. Secondly, we will use the theory of Bäcklund transformations to understand the symmetries of a given curve. All the computations will be carried out in the simplest case of planar circles, where interesting results already arise.

4.1. **The Bäcklund formula for the Heisenberg Model.** We first derive the Bäcklund transformation for the HM equation and the expression for the corresponding transformed curve [8]. To this end, we use the Bäcklund formula (69) and the expression of the gauge transformation between the HM and NLS.

Proposition 4.1 (HM Bäcklund Transformation)**.** *Let $\boldsymbol{\phi} = c_+\boldsymbol{\phi}_+ + c_-\boldsymbol{\phi}_-$ be a complex linear combination of linearly independent solutions of the Lax pair (24) at (S, ν). We construct the matrix of gauge transformation*

$$G(\lambda, \nu, \boldsymbol{\phi}) = N \begin{pmatrix} \frac{\nu-\lambda}{\nu} & 0 \\ 0 & \frac{\bar{\nu}-\lambda}{\bar{\nu}} \end{pmatrix} N^{-1} \tag{71}$$

with

$$N = \begin{pmatrix} \phi_1 & -\bar{\phi}_2 \\ \phi_2 & \bar{\phi}_1 \end{pmatrix}. \tag{72}$$

Then, if $\mathbf{F}$ *solves the linear system (24) at (S, λ), the vector*

$$\mathbf{F}^{(1)}(x, t; \lambda, \nu) = G(\lambda, \nu, \boldsymbol{\phi})\mathbf{F}(x, t; \lambda) \tag{73}$$

solves (24) at $(S^{(1)}, \lambda)$ and the new solution of the HM equation is given by the Bäcklund formula

$$S^{(1)}(x, t) = N \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} N^{-1} S(x, t) N \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} N^{-1} \tag{74}$$

with $e^{i\theta} = \dfrac{\nu}{|\nu|}$.

Remarks. 1. The matrix $U = N \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} N^{-1}$ is unitary, hence the solution $S^{(1)}$ is hermitian and has zero trace (and its determinant is 1). The corresponding vector $\mathbf{T}^{(1)}$ is therefore obtained by a rotation of the tangent vector $\mathbf{T}$ of the original curve by a constant angle $\theta = \frac{\nu}{|\nu|}$ depending only on the eigenvalue parameter ν, around the instantaneous axis of rotation

$$\left(\frac{2Re(\phi_1 \bar{\phi_2})}{|\phi_1|^2 + |\phi_2|^2}, -\frac{2Im(\phi_1 \bar{\phi_2})}{|\phi_1|^2 + |\phi_2|^2}, \frac{|\phi_1|^2 - |\phi_2|^2}{|\phi_1|^2 + |\phi_2|^2} \right)^T.$$

2. If $\nu \in \mathbb{R}$, then $S^{(1)} = S$ and the Bäcklund transformation is the identity. If ν is chosen such that the corresponding eigenfunction $\boldsymbol{\phi}$ is periodic or antiperiodic in the x-variable (such ν is called a periodic/antiperiodic eigenvalue), then the corresponding solution is periodic (i.e. the transformed curve is closed). For any other choice of $\nu \in \mathbb{C}$, formula (74) produces quasi-periodic solutions of the nonlinear equation.

Proof. [8] The gauge formula allows us to write the eigenfunction of the NLS Lax pair at (q, λ) in terms of the solution of the HM linear system at (S, λ) as $\mathbf{F}_{\mathrm{NS}} = V\mathbf{F}$, where V is the gauge matrix introduced in section 1.5. The eigenvector $\mathbf{F}^{(1)}_{\mathrm{NS}}$ of the NLS Lax pair at $(q^{(1)}, \lambda)$ can similarly be expressed as $\mathbf{F}^{(1)}_{\mathrm{NS}} = V^{(1)}\mathbf{F}^{(1)}$, where $V^{(1)}$ is the new gauge matrix. Since $V^{(1)}$ solves the linear system for NLS at $(q^{(1)}, 0)$, it can be written as $V^{(1)} = G_{\mathrm{NS}}|_{\lambda=0}V$ and we can write the new eigenfunction for the HM

$$\mathbf{F}^{(1)} = V^{-1} \left(G_{\mathrm{NS}}|_{\lambda=0} \right)^{-1} G_{\mathrm{NS}} V\mathbf{F}. \tag{75}$$

We make the important observation that the expression of $\mathbf{F}^{(1)}$ is independent of the gauge transformation, and therefore it does not depend on the NLS eigenfunctions. In order to show this we use the fact that V is a unitary matrix with determinant 1, and compute

$$V^{-1} N_{\mathrm{NS}} = V^{-1} \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix} = \begin{pmatrix} \phi_1 & -\bar{\phi}_2 \\ \phi_2 & \bar{\phi}_1 \end{pmatrix} = N, \tag{76}$$

with $\boldsymbol{\psi} = V\boldsymbol{\phi}$ and $\boldsymbol{\phi} = c_+\boldsymbol{\phi}_+ + c_-\boldsymbol{\phi}_-$ a complex linear combination of independent eigenfunctions of the linear system (24). Using (76) to simplify equation (75) we obtain expression (73) for the new eigenfunction. Substituting the expression for $\mathbf{F}^{(1)}$ in the equation

$$\frac{\partial}{\partial x}\mathbf{F}^{(1)} = i\lambda S^{(1)}\mathbf{F}^{(1)}, \tag{77}$$

we derive the following formula for the transformed potential

$$S^{(1)} = \frac{1}{i\lambda}\frac{\partial G}{\partial x}G^{-1} + GSG^{-1}. \tag{78}$$

The dependence on λ is only apparent, by computing the x-derivative of G and rearranging terms we can reduce (78) to the form of equation (74). $\square$

In order to derive the formula for the corresponding curve in space, we use the representation of the curve in terms of the fundamental solution of the linear problem.

$$\Gamma_{\mathrm{new}} = -i \left.\frac{\partial}{\partial \lambda}\right|_{\lambda=0} \Phi_{\mathrm{new}}\Phi_{\mathrm{new}}^{-1}|_{\lambda=0} + C, \tag{79}$$

where C is a hermitian x-independent. Using expression (73) for Φ_{new}, we find (up to an x-independent translation)

Corollary 3. *The Bäcklund transformation for the filament equation is described by the following formula (in matrix representation):*

$$\Gamma_{new} = \Gamma + \frac{Im(\nu)}{|\nu|^2}V \tag{80}$$

with

$$V = N\sigma_3 N^{-1} \tag{81}$$

Formula (80) reduces to the identity when the eigenvalue ν is real, and it produces bounded solutions of the filament equation (not necessarily periodic in the space variable x) for any complex ν. In fact, directly from the formula for the new curve we deduce the following confinement result,

Proposition 4.2.

$$||\Gamma|| - \frac{|Im(\nu)|}{|\nu|^2} \leq ||\Gamma_{new}|| \leq ||\Gamma|| + \frac{|Im(\nu)|}{|\nu|^2} \tag{82}$$

($||\cdot||$ is the standard Euclidean norm in $\mathbb{R}^3$). The transformed curve is confined to the interior of a sphere if $\dfrac{|Im(\nu)|}{|\nu|^2} \geq \max_x ||\Gamma(x)||$ *or to the interior of a torus if* $\dfrac{|Im(\nu)|}{|\nu|^2} < \max_x ||\Gamma(x)||$.

Sym et al. [11] obtained this same confinement result (which is of interest in the context of vortex dynamics) by reconstructing the curve from a basis of eigenfunctions of the NLS Lax pair and by using directly the NLS Bäcklund transformation.

4.2. **An application to planar circles.** The simplest solutions which possess homoclinic instabilities are planar circles. Their tangent vectorfields are fixed points of the Continuous Heisenberg Model, therefore the expression of the associated curve can be chosen to be time-independent. We consider k copies of a circle lying in the (x, y)-plane with non zero curvature $k \in \mathbb{Z}$ (an integral curvature is required for the circle to be closed). If x denotes the arclength parameter, the hermitian matrix which represents the unit tangent vector is

$$S_0(x,t) = \begin{pmatrix} 0 & e^{-ikx} \\ e^{ikx} & 0 \end{pmatrix} \tag{83}$$

Following [16, 17, 18], we make use of the Floquet spectrum of the linear operator

$$\mathcal{L}(S,\lambda) = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx} + i\lambda S \tag{84}$$

(the spatial part of the HM Lax pair) in order to characterise the level set on which S_0 resides. The fundamental solution matrix $\Phi(x, y; S, \lambda)$ is defined by the following properties

$$\begin{aligned} &\mathcal{L}(S,\lambda)\Phi = 0, \\ &\Phi(x,x;S,\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{85}$$

We introduce the *transfer matrix* $\Phi(x + 2\pi, x; S, \lambda)$ as the transformation which takes a solution of (84) evaluated at x to its value after a period 2π. It is easy to check that the transfer matrix has determinant 1, as a consequence its eigenvalues are completely characterised in terms of its trace

$$\Delta(S,\lambda) = \mathrm{Trace}\left[\Phi(x+2\pi,x;S,\lambda)\right]. \tag{86}$$

The quantity $\Delta(S, \lambda)$ is called the Floquet discriminant of the linear operator $\mathcal{L}$. Recalling that the spectrum of a linear operator is the set of eigenvalues for which the corresponding eigenfunction is bounded in x, then the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ is the set of λ's for which the eigenvalues of the transfer matrix lie on the unit circle. In terms of the trace of the Floquet discriminant,

$$\sigma(\mathcal{L}) = \{\lambda \in \mathbb{C} \,|\, \Delta(S,\lambda) \in \mathbb{R} \text{ and } -2 \le \Delta(S,\lambda) \le 2\}.$$

The discriminant has the following properties (property 1 follows from the analyticity of the fundamental solution of the linear system, the verification of property 2 is a simple computation):

Theorem 4. *If S solves the Continuous Heisenberg Model then,*

1. *$\Delta(S, \lambda)$ is an analytic function of λ.*
2. *$\Delta(S, \lambda)$ is invariant under the evolution, i.e.*

$$\frac{d}{dt}\Delta(S(t),\lambda) = 0. \tag{87}$$

Remark. Since the discriminant is a constant of motion which depends analytically on the complex parameter λ, Δ provides us with a 1-complex parameter family of invariants. In fact, one can extract the usual hierarchy of constant of motions from the coefficients of the asymptotic expansion of the discriminant at a distinguished value of λ [20]. Moreover, we can characterise the level set on which a given solution S_0 resides as its isospectral set (the set of all potentials which have the same spectrum as S_0).

We introduce the following distinguished points of $\sigma(\mathcal{L})$:

critical points λ_c:

$$\frac{d}{d\lambda}\Delta(S,\lambda)\Big|_{\lambda=\lambda_c} = 0,$$

periodic (antiperiodic) points $\lambda_\pm$:

$$\Delta(S,\lambda)|_{\lambda=\lambda_\pm} = \pm 2,$$

multiple points λ_m:

$$\begin{aligned} \Delta(S,\lambda)|_{\lambda=\lambda_m} &= \pm 2 \\ \frac{d}{d\lambda}\Delta(S,\lambda)\Big|_{\lambda=\lambda_m} &= 0 \end{aligned}$$

The periodic (antiperiodic) points are associated with periodic (antiperiodic) eigenfunctions. The Bäcklund transformation at one such point produces a solution which is periodic in x. The existence of complex multiple points (we will consider only double points, i.e. critical points of multiplicity 2) is related to the presence of linear instabilities. The level set of a solution whose spectrum contains complex double points is saddle-like and the corresponding homoclinic orbits can be constructed by means of Bäcklund transformations. We refer the reader to references [16, 17, 18] for the discussion of these general results. We will instead infer these conclusions for the simplest example of planar circles and their associated Bäcklund transformations.

The HM Lax pair can be easily solved when S is the matrix S_0 representing a multiple covered circle as in (83). The associated Floquet discriminant is

$$\Delta(S_0,\lambda) = \Delta(k,\lambda) = (-1)^k 2\cos(\delta\pi) \quad \text{with} \quad \delta = \sqrt{k^2+4\lambda^2},$$

and the corresponding spectrum is (see figure 4.2)

$$\sigma(S_0,\lambda) = \{\lambda \in \mathbb{C} \mid \cos(\delta\pi) \in \mathbb{R}\} = \mathbb{R} \cup \{i\mu \mid \mu \in \mathbb{R}\ ,\ -k/2 \le \mu \le k/2\}.$$

The critical points are the set of solutions of

$$\frac{d}{d\lambda}\Delta = -\sin(\delta\pi)\frac{4\lambda}{\sqrt{k^2+4\lambda^2}} = 0.$$

All of them are double points except for $\lambda = 0$ which is a critical point of multiplicity 4. There is a finite number of complex double points (in this case they are purely imaginary) given by

$$\lambda_n = \pm\frac{i}{2}\sqrt{k^2-n^2} \qquad n = 1, 2, \dots, k-1.$$

We now implement the Bäcklund transformation derived in the previous section

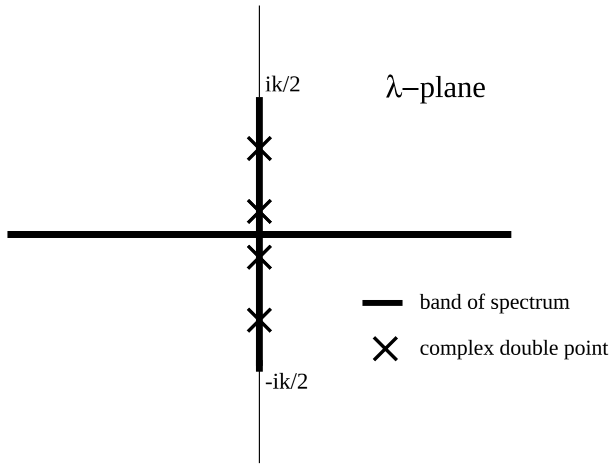


FIGURE 3. The spectral configuration of S_0

and construct the corresponding new solutions of the HM. We choose two linearly independent solutions of the linear problem at (S_0, ν) as follows,

$$\boldsymbol{\phi}_- = e^{-\frac{i}{2}\delta(x+2\nu t)} \begin{pmatrix} -2\frac{\nu}{k+\delta} e^{-\frac{i}{2}kx} \\ e^{\frac{i}{2}kx} \end{pmatrix}, \quad \boldsymbol{\phi}_+ = e^{\frac{i}{2}\delta(x+2\nu t)} \begin{pmatrix} e^{-\frac{i}{2}kx} \\ \frac{\delta-k}{2} e^{\frac{i}{2}kx} \end{pmatrix}.$$

$\boldsymbol{\phi}_+$ and $\boldsymbol{\phi}_-$ are called Baker eigenfunctions and are uniquely characterised by their expression in terms of an exponential phase factor multiplied by a bounded (periodic) vector function.

We distinguish two classes of Bäcklund transformations which generate qualitatively different solutions in the level set of S_0:

Type I Bäcklund Transformation. If we take $\boldsymbol{\phi}$ in formula (80) to coincide with one of the Baker eigenfunctions at the eigenvalue ν, we obtain a periodic solution for every choice of $\nu \in \mathbb{C}$. Moreover such solution belongs to the same class S_0; in particular, for $\boldsymbol{\phi} = \boldsymbol{\phi}_+$ we obtain the following family of planar circles

$$\begin{aligned} \Gamma_1(x,\nu) = &\begin{pmatrix} 0 & \frac{i}{k}e^{-ikx} \\ -\frac{i}{k}e^{ikx} & 0 \end{pmatrix} + \\ &\frac{Im(\nu)}{|\nu|^2(4|\nu|^2 + |\delta - k|^2)} \begin{pmatrix} 4|\nu|^2 - |\delta - k|^2 & 4\nu(\bar{\delta} - k)e^{-ikx} \\ 4\bar{\nu}(\delta - k)e^{ikx} & 4|\nu|^2 - |\delta - k|^2 \end{pmatrix} \end{aligned} \tag{88}$$

parametrised by the complex parameter ν.

Remark. In order to interpret the expression of the new curve as the effect of the action of a symmetry, we compute an infinitesimal Bäcklund transformation as $Im(\nu) \to 0$. In this limit, $\Gamma_1 - \Gamma_0$ approximates a family of vectorfields along the original curve Γ_0 of the form

$$\mathbf{V}(x,\nu) = c_1(\nu)(0,0,1)^T + c_2(\nu)\mathbf{S}_0. \tag{89}$$

where c_i's are constants depending on ν. Formula (89) describes a family of Killing vectorfields (infinitesimal isometries) for the the original circle: the sum of a constant translation field and a rotation. Already in this simple example we see that a type I Bäcklund transformation realises the symmetries of the level set of the solution S_0. Such symmetries manifest themselves at the curve level as infinitesimal isometries.

Type II Bäcklund transformation. We now take $\phi = c_-\phi_- + c_+\phi_+$ to be a general complex linear combination of Baker eigenfunctions. In this case, the transformed curve is closed if ν is a periodic/antiperiodic eigenvalue. When ν is chosen to be one of the complex double points $\nu = i\mu_n = \pm\frac{i}{2}\sqrt{k^2-n^2}$, $n = 1, \dots, k-1$, setting $c_+/c_- = \rho e^{i\theta}$, we obtain the following formula:

$$\Gamma_2(x,t;\mu_n) = \begin{pmatrix} 0 & \frac{i}{k}e^{-ikx} \\ -\frac{i}{k}e^{ikx} & 0 \end{pmatrix} + \frac{1}{\mu_n}\begin{pmatrix} f(x,t)-f(0,t) & \overline{g(x,t)}-\overline{g(0,t)} \\ g(x,t)-g(0,t) & -(f(x,t)-f(0,t)) \end{pmatrix},$$

where

$$\begin{aligned} f(x,t) &= -\frac{1}{2\mu_n}\frac{h'(t)}{kh(t)+8\rho\mu_n^2\sin(nx+\theta)}, \\ g(x,t) &= -2i\frac{\mu_n}{k}e^{-ikx} - 4\,\rho n\mu_n\frac{\cos(nx+\theta)+i\frac{n}{k}\sin(nx+\theta)}{kh(t)+8\rho\mu_n^2\sin(nx+\theta)}e^{-ikx}, \\ h(t) &= [k+n+(k-n)\rho^2]\sinh(n\mu t) + [k+n-(k-n)\rho^2]\cosh(n\mu t). \end{aligned}$$

Since $\lim_{t\to\pm\infty}\Gamma_2(x,t) = \Gamma_0(x)$, the new curve Γ_2 is a homoclinic orbit for the original solution, (this also implies that multiple covers of circles are unstable solutions of the filament equation). Moreover we conclude that Γ_2 resides on the same level set as Γ_0, indeed spectral configurations such as the one depicted in Figure 3 describe both the target solution and its homoclinic orbits.

Figure 4 shows a sequence of time frames of the evolution of the curve Γ_2 homoclinic to a 6-fold circle. The new curve possesses 5 points of self-intersection which persist throughout the dynamics. It appears to be a general feature of these singular curves that the number of such "stable" self-intersections equals the ordering of the complex double point.

An interesting question is whether the presence of self-intersections has a topological meaning. A more general related question is whether the curves produced by a type II Bäcklund transformation play a role in distinguishing different knot classes of multiphase solutions, and whether the Floquet spectrum can be used to characterise topological invariants. Figures 5b, 5c show typical spectral configurations of initial data close to a planar circle with a single complex double point (Figure 5a). The spectra can be computed using standard perturbation analysis (we consider only spatially even solutions, the general case being far more complicated [19]). The degenerate double point splits in a "cross" (b) or a "gap" (c) configuration characteristic of different 3-phase solutions which in turn correspond to complicated, likely knotted curves in space. Figure 5 illustrates the role of the homoclinic orbit to the planar circle as a separatrix between different types of 3-phase solutions, how this should be interpreted at the curve level and whether this fact carries a topological significance is an open problem being currently investigated by the author. A positive answer would support the use of the Floquet spectrum in the topological classification of N-phase curves.

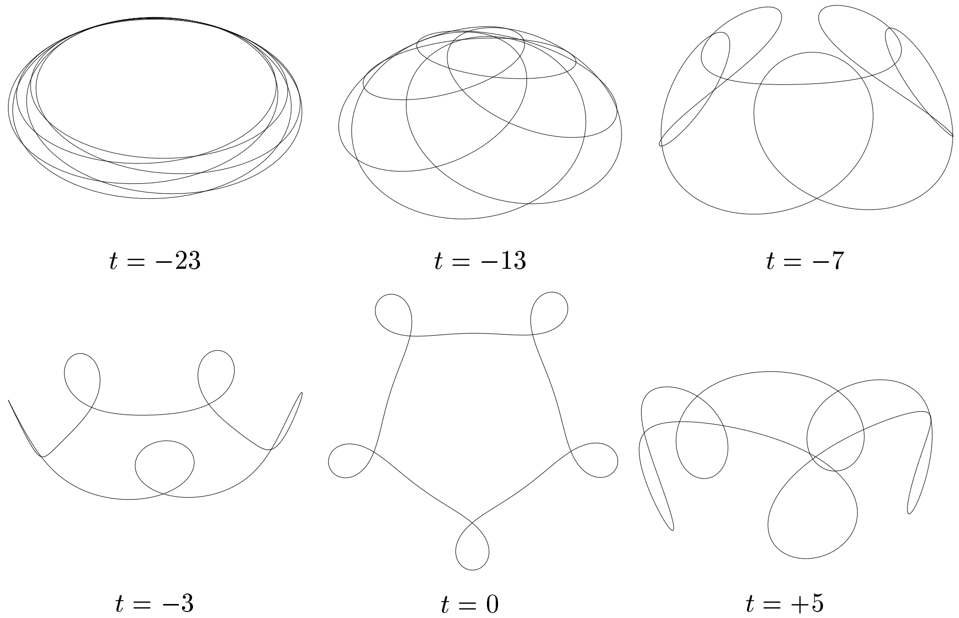


FIGURE 4. Evolution of the Bäcklund transformation of a 6-fold planar circle. $k = 6$, $n = 5$, $\rho = 1$, $\theta = 0$

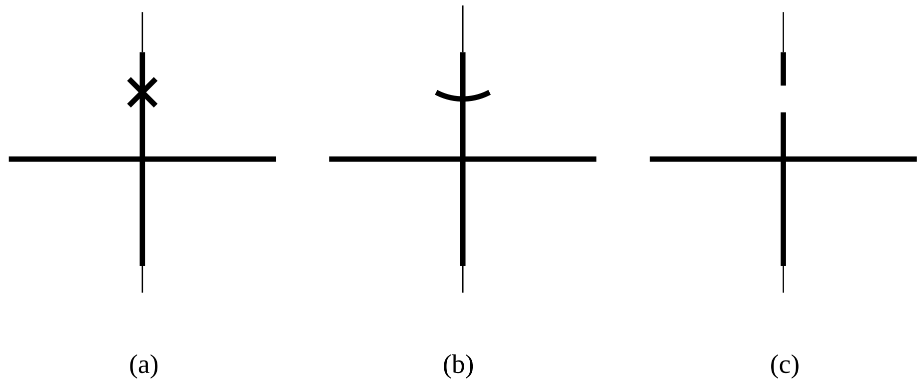


FIGURE 5

5. FUTURE DIRECTIONS

We conclude with a description of current research activities and a list of open questions.

To my knowledge, the following groups have been working on related topics: A. Sym et al. (see e.g. [11, 54, 53] have worked on applications of soliton theory to vortex filament dynamics and on the characterisation of soliton surfaces generated by the evolution of vortex filaments in the shape of solitons. H. K. Moffat, R. L. Ricca and collaborators (at Cambridge and University College, London) have focused on applications of knot theory to fluid mechanics (see e. g. [49, 47, 48]. The work of J. P. Keener and J. J. Tyson [29, 28] on scroll waves in excitable media and on vortex filaments in ideal fluids has important points of contacts with the author's work.The group at Case Western Reserve University including J. Langer, D. Singer, T. Ivey and myself, together with collaborator R. Perline at Drexel University, have worked on higher-dimensional generalisations of the filament equation hierarchy and on the topological implications of related Bäcklund transformations. A. Doliwa and P. M. Santini have addressed similar questions about integrable curve evolutions [13] on n-dimensional spheres and have also discussed the discrete problem in the context of polygonal spherical curves [14]. On the same theme of polygonal curves, J. Millson at the University of Maryland and M. Kapovich at Utah State University [27] have studied the topology of the space of polygonal knots, its symplectic structure and an associated integrable evolution. The continuous counterpart has been studied by J. Millson and B. Zombro [43], with a focus on the symplectic geometry of smooth curves in Euclidean space and on the bihamiltonian structure of the periodic Heisenberg Model.

There are several open directions of research.

1. The formulation and study of more realistic models of vortex filament dynamics which take account of the finite structure of the vortex core, the longitudinal stretching of the filament, nonlocal interactions and the effect of finite viscosity.
2. Discrete models of integrable curve dynamics (i.e. evolution equations of polygonal curves): discrete soliton equations such as the Ablowitz-Ladik model or integrable discretisations of the Heisenberg Chain are among them.
3. Other integrable hierarchies are associated to curve dynamics, among them the KdV, mKdV and the Sine-Gordon equations: a natural question is whether any soliton equation can be associated with a curve evolution.
4. Closely related to the previous question is the construction of a general framework in which to describe integrable hierarchies of curve dynamics in a space of arbitrary dimension; a suitable set-up appears to be the Kac-Moody Lie algebra framework where a soliton equation is regarded as a dynamical system on an infinite-dimensional loop algebra. Since this work was written Langer and Perline have shown that dynamical systems associated with Hermitian symmetric Lie algebras can be used to model the evolution of arclength parametrized curves in Euclidean and other spaces, see [36] for details.
5. In the context of higher phase solutions of the filament equation, what do the higher N-phase curves look like? is it possible to use the Floquet spectrum of the spatial part of the associated Lax pair to classify their knot type? The author is currently working on implementing the general expression for N-phase curves in terms of Riemann theta functions discussed in this work. Recently, in an interesting article by P. G. Grinevich and M. U. Schmidt [25], conditions for the closure of the corresponding curves in terms of the NLS spectral problem are derived (generalizing the relation between complex double points and close Bäcklund transformations that we explicitly computed in the elementary case of planar circles).
6. Also, does the Bäcklund transformation carry any topological information? A partial answer has been provided (since the date of this article) by the author and her collaborator T. Ivey [9] with respect to closed constant torsion elastic rods, that can be associated with the Sine-Gordon equation. Bäcklund transformations are shown to provide a variety of topological phenomena such as knotting, unknotting and self-intersections. Questions about the mechanisms of topological changes are

gaining attention in a variety of fields (e.g. [41]).
7. Moffat and Ricca posed an analogous question for the sequence of global geometrical invariants: whether they restrict the possible knot types of the corresponding curves and whether they themselves have a topological interpretation.
8. The Marsden-Weistein Poisson structure is realised naturally at the level of curves as a 90-degree rotation of any normal vectorfield to the curve and it is associated to the almost-complex structure on the loop space of $\mathbb{R}^3$: an open question is whether the higher Poisson structures have a natural geometrical realisation, and whether the zero-curvature formulation of the sequence of NLS flows (and of general soliton equations) can be given a direct geometric interpretation.

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