TOPOLOGY AND SINE-GORDON EVOLUTION OF CONSTANT TORSION CURVES

ANNALISA M. CALINI
Dept. of Mathematics, College of Charleston, Charleston, SC 29424
calini@math.cofc.edu
and
THOMAS A. IVEY
Dept. of Mathematical Sciences, Ball State University, Muncie IN 47306
tivey@math.bsu.edu

ABSTRACT. The sine-Gordon equation with periodic boundary conditions describes integrable dynamics on the space of closed curves of constant torsion, for which multi-phase solutions provide large classes of canonical knot representatives. In this letter we discuss the use of Bäcklund transformations for studying the topological properties and symmetries of multi-phase solutions and of their homoclinic manifolds.

1. INTRODUCTION

In a series of studies of the geometry of the periodic sine-Gordon equation

$$u_{st} = \sin u,$$

N. Ercolani, G. Forest and D. McLaughlin [5, 6] describe the structure of the class of multi-phase solutions and their associated hyperbolic invariant manifolds. The main tools used are: the Floquet theory [10] for the spatial part of the associated Lax Pair

$$\frac{d\psi}{ds} = \frac{1}{2}\begin{pmatrix} \lambda & u_s \\ -u_s & -\lambda \end{pmatrix}\psi, \qquad \frac{d\psi}{dt} = \frac{1}{2\lambda}\begin{pmatrix} \cos u & -\sin u \\ -\sin u & -\cos u \end{pmatrix}\psi, \tag{1}$$

in terms of which both an implicit representation of the isospectral set of a given N-phase solution and its linear stability type can be obtained; and, Bäcklund transformations (built from solutions of (1)) which, when iterated, give an explicit representation of both the isospectral set and its homoclinic manifold.

On the other hand, the sine-Gordon equation and its Bäcklund transformation are well-known to differential geometers. Solutions of the sine-Gordon equation are in one-to-one correspondence with surfaces of constant negative Gaussian curvature (i.e., pseudospherical surfaces), and the Bäcklund transformation can be interpreted geometrically as a line congruence [4] connecting a given pseudospherical surface to a new one. The geometrical applications of the sine-Gordon equation also extend to space curves: if $\Psi(s;\lambda)$ is the fundamental solution matrix of the spatial part of the sine-Gordon Lax pair (1), the $su(2)$ matrix

$$\gamma(s,t) = i\Psi^{-1}\left.\frac{d\Psi}{d\lambda}\right|_{\lambda=-i\tau} \tag{2}$$

can be identified with a space curve of constant torsion τ and curvature $\kappa = u_s$, if we fix an isometry from $su(2)$ to $\mathbb{R}^3$. Furthermore, the Bäcklund tranformation for a pseudospherical surface can be restricted to one of its asymptotic lines to obtain a Bäcklund transformation on the space of constant torsion curves [3].

In [3] we study the geometrical and topological properties of a family of closed constant torsion curves, associated to travelling wave solutions of the sine-Gordon equation, and their Bäcklund transformations. These curves, which arise as the centerlines of elastic rods [7], realize a well-known class of knot types: for every pair of relatively prime integers (m, n) with $1 < |m| < n/2$, there is a unique (m, n) torus knot among these curves. The evolution of such curves by sine-Gordon depends on two distinct phases, one of which corresponds to a shift of curvature along the curve, and the other to a rigid motion in space. Bäcklund transformations of two kinds are defined in [3], referred to as *single* and *double*. The former is shown to reflect (via linking numbers) the topology of these curves, while the latter is used to produce constant torsion realizations of a large number of additional knot types.

The present letter contains several results that generalize and complement those in [3]. In §2 the linking number of a general closed constant torsion curve and its single Bäcklund transformation is computed in terms of the geometry of the former. (This result generalizes an earlier formula obtained only for 2-phase solutions.)

In §3 we remark that an infinitesimal Bäcklund transform of a 2-phase curve is the restriction of a slide-Killing field along the curve. Iterations of Bäcklund transformations built from solutions of (1) at real points of the Floquet spectrum generate the whole isospectral set of a given N-phase solution [6]; thus, infinitesimal transformations generate the tangent space, including the symmetry vector fields, of the level set. We conjecture that, in the curve context, the corresponding vector fields are restrictions of slide-Killing fields along any given N-phase solution.

In §4 we discuss the structure of Bäcklund transformations iterated at a pair of complex conjugate double points of the Floquet spectrum. We provide a formula for the number of complex double points of the Floquet discriminant associated to a closed 2-phase curve, and exhibit the associated Floquet spectrum. Complex double points are generically associated to linear instabilities [5]; in this case the Bäcklund formula will give an explicit representation of the manifold of homoclinic orbits [6].

In the final section the time evolution is explicitly constructed for the 2-phase curves and their homoclinic orbits. The time-dependence for solutions to sine-Gordon and to (1) is deduced by formulating the time vector field in terms of the Killing fields along the "seed" curve. Besides showing typical pseudospherical surfaces generated by the time evolution of homoclinic orbits of constant torsion 2-phase curves, we show that the evolving curve may undergo several topological changes, i.e., the knot type is not preserved by the integrable dynamics. This is in contrast to the behaviour of Bäcklund transformations of one-phase solutions (translating circles) of a related flow, the Localized Induction Equation; in [2], examples are computed of closed curves which are asymptotic to the circle, but feature self-intersections that are preserved by the flow.

2. Linking and the single Bäcklund transformation

The *single Bäcklund transformation* for constant torsion curves can be formulated in terms of a gauge transformation of the fundamental matrix solution $\Psi(s;\lambda)$ of the spatial part of the sine-Gordon linear system (1),

$$\frac{d\psi}{ds} = \frac{1}{2}\begin{pmatrix} \lambda & \kappa \\ -\kappa & -\lambda \end{pmatrix}\psi. \tag{3}$$

Here, u_s is set equal to the Frenet curvature κ of the curve γ, as is the case for asymptotic lines on a pseudospherical surface. Given a vector solution $\psi(s)$ of (3) at $\lambda = \nu \in \mathbb{R}$, one lets

$$\Psi^{(1)} = G(s;\lambda,\nu)\Psi, \tag{4}$$

where

$$G = \frac{1}{\sqrt{\lambda^2-\nu^2}}\left(\lambda I + \nu\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}\right), \tag{5}$$

with $\beta = -2\arctan(\psi_1/\psi_2)$. Then the new curve γ_ν is obtained by replacing Ψ by $\Psi^{(1)}$ in formula (2); $\Psi^{(1)}$ now solves system (3) when κ is replaced by the curvature of γ_ν. If (T,N,B) is the Frenet frame of γ, then

$$\gamma_\nu = \gamma + \frac{2\nu}{\nu^2+\tau^2}(T\cos\beta + N\sin\beta). \tag{6}$$

Note that β satisfies the differential equation

$$d\beta/ds = \nu\sin\beta - \kappa. \tag{7}$$

In this section we want to derive a formula for the linking number of a closed constant torsion curve γ and its closed Bäcklund transform γ_ν; this will be computed in terms of the curvature and self-linking number of γ. Our main tool will be White's formula [9, 11].

Assume that γ is closed of length L, and has a closed Bäcklund transformation γ_ν for all ν sufficiently close to zero,[1] defined by an angle $\beta(s;\nu)$ depending analytically on ν. Let γ_ν have the orientation it inherits from γ under (6).

Let $V = T\cos\beta + N\sin\beta$. The ribbon between γ and $\gamma+\delta V$ will be embedded for δ sufficiently small, except at points where V is tangent to γ. For simplicity, assume this does not happen where $\kappa = 0$; we'll also assume $\tau > 0$. Consider the perturbed vector

$$\tilde{V} = V - \epsilon(\cos\beta)\kappa B, \qquad \epsilon > 0$$

As noted in [3], $\gamma + \delta\tilde{V}$ is an embedded ribbon for ϵ sufficiently small.

[1] This is satisfied if the eigenvalues of the fundamental matrix of (3) at $\lambda = 0, s = L$ are real and distinct.

Let U be the unit vector in the direction of the projection of $\tilde{V}$ onto the plane normal to T. Then White's formula gives

$$Lk(\gamma, \gamma_\nu) = Wr(\gamma) + \frac{1}{2\pi} \int (T \times U) \cdot dU,$$

where the writhe Wr is related to the self-linking number by Pohl's formula [8]

$$SL(\gamma) = Wr(\gamma) + \frac{1}{2\pi} \int \tau ds. \tag{8}$$

Ordinarily, $SL(\gamma)$ is defined as the linking number of γ and $\gamma + \delta N$. If γ is an odd curve, where κ and the Frenet frame vectors N and B have antiperiod L, then we take (8) as the definition of $SL(\gamma)$. Then, $SL(\gamma)$ is either an integer or a half-integer.

Further calculation gives

$$Lk(\gamma, \gamma_\nu) = SL(\gamma) + \frac{1}{2\pi} \int d \left(\arctan \left(\frac{\tan \beta}{\epsilon \kappa} \right) \right)$$

The last term contributes $+1/2$ (resp. $-1/2$) for every interval on which $\tan \beta / \kappa$ passes from $-\infty$ to $+\infty$ (resp. $+\infty$ to $-\infty$). However, (7) shows that whenever $\sin \beta$ changes sign, $\tan \beta / \kappa$ goes from positive to negative. Let j be the number of times $\sin \beta$ changes sign; then

$$Lk(\gamma, \gamma_\nu) = SL(\gamma) - j/2.$$

Since this is an equation of integers (or half-integers), our extra assumption about the transversality of V at inflection points can be removed by adjusting ν. Furthermore, since torsion, linking and self-linking numbers change sign under reflections, our final formula is

$$Lk(\gamma, \gamma_\nu) = SL(\gamma) - j/2 \operatorname{sgn}(\tau). \tag{9}$$

The integer j can be calculated from the curvature function κ alone. Suppose the angle β giving a closed curve has a series expansion

$$\beta(s; \nu) = \beta_0(s) + \beta_1(s)\nu + O(\nu^2).$$

Substitution in (7) gives

$$\beta_0' = -\kappa, \qquad \beta_1' = \sin \beta_0.$$

Since $\int_0^L \beta' ds = \int_0^L \beta_0' ds$ by continuity, $\int_0^L \beta_1' ds = 0$. Hence β_0 is an antiderivative of $-\kappa(s)$ such that $\int_0^L \sin(\beta_0) ds = 0$, and j is the number times $\sin(\beta_0)$ changes sign.

Of course, this can also be calculated using an antiderivative θ of $+\kappa(s)$. Then the vector $(\cos \theta, \sin \theta)$ is the unit tangent for a planar curve with curvature κ, rotated so that it starts and ends on the x-axis, and j is the number of points along this curve where the tangent is parallel to the straight line between the endpoints at $s = 0$ and $s = L$. (If κ has antiperiod L, j is calculated by taking $s = 0$ and $s = 2L$ as endpoints and dividing this count by 2.) Of course, rotation is not necessary to calculate j.

The linking number formula (9) is illustrated in Figures 1 and 2.

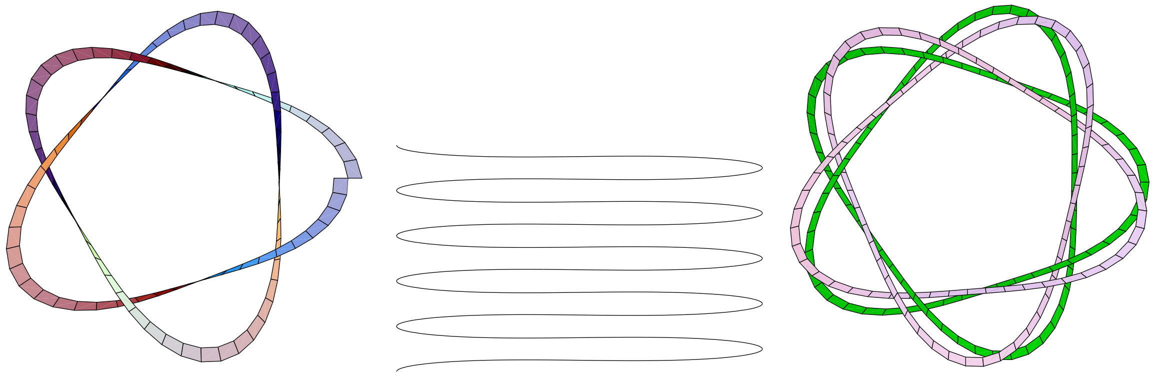

FIGURE 1. At left is the ribbon formed by the normal vector of the knotted (2,5) elastic rod of positive constant torsion, oriented clockwise. (Since the curvature is only periodic up to a minus sign, the ribbon does not close up.) Self-linking, calculated by counting the crossings of the outer edge of the ribbon with the centerline, is $15/2$. In the middle is a planar curve with the same curvature function (and twice the length) as the rod. Counting points where the tangent line is parallel to the line segment between the ends gives $j = 10/2 = 5$. At right is the rod (darker ribbon) and one of its nearby closed Bäcklund transformations, showing that the linking number of the two is 5, in agreement with our formula.

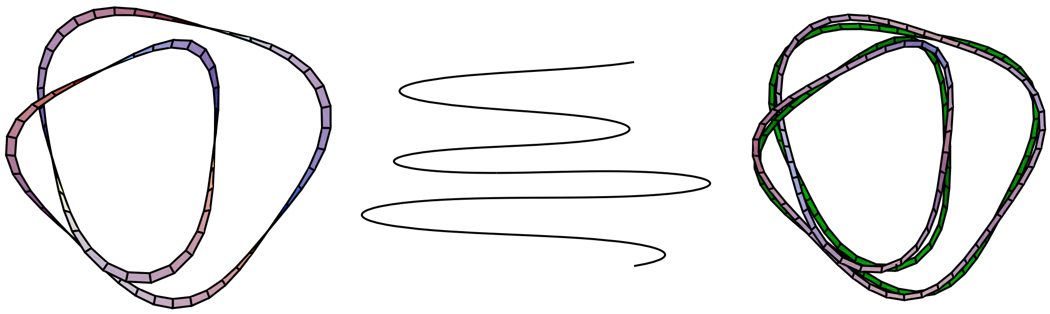

FIGURE 2. At left is the normal ribbon for a double Bäcklund transformation of the unknotted (1,3) elastic rod of positive torsion, showing self-linking is 4. The middle figure shows $j = 6$. At right is the curve and one of its nearby closed single Bäcklund transformations; the linking number is 1, agreeing with our formula.

3. A Remark on Killing Fields

Iterations of Bäcklund trasnformations at real points of the spectrum of a given N-phase solution are known to generate its entire isospectral set [6], thus leaving the vectors of frequencies and wave numbers of the solution of sine-Gordon unchanged. At the curve level we conjecture that Bäcklund transformations at these real points are associated to the presence of Killing fields (i.e., restrictions of vector fields which generate a one-parameter group of rigid motions in the ambient space) along the curve. More precisely, iterated infinitesimal Bäcklund trasformations exhaust the generators of the slide-Killing fields along the given curve.

We prove this for an elastic rod of constant torsion. In the case of elastic rods, Langer and Singer [7] showed that the only Killing fields are linear combinations of the screw field $V_1 = 4\tau T + 2\kappa B$ and the translation field $V_2 = (\kappa^2 - c)T + 2\kappa_s N - 2\kappa\tau B$, where c is a real constant. As the spectral parameter $\nu \to 0$, we compute

$$\gamma_\nu - \gamma = \frac{2\nu}{\tau^2}\,(T\cos\beta + N\sin\beta) + O(\nu^2),$$

where the variable β satisfies $d\beta/ds = -\kappa + O(\nu)$. We show that $X = T\cos\beta + N\sin\beta$ is slide-Killing, i.e. it differs from a Killing field by a constant multiple of the tangent vector.

A vector field Y is Killing if the Y-derivatives of the curve speed, curvature and torsion vanish. In [7] the following expressions for the variations of the speed v, curvature κ and torsion τ are given:

$$Y(v) = vT\cdot Y_s, \qquad Y(\kappa) = N\cdot Y_{ss} - 2\kappa\tau Y_s, \qquad Y(\tau) = [(B/\kappa)\cdot Y_{ss}]_s + (\kappa B - \tau T)\cdot Y_s.$$

For the vector field $Y = \mu T + X$, where μ is a constant to be determined, we find $Y(v) = Y(\tau) = O(\nu)$ and $Y(\kappa) = \mu\kappa_s - \tau^2\sin\beta + O(\nu)$. We only need to show that these vanish up to $O(\nu)$.

For a closed elastic rod of constant torsion, the expression of the curvature function is $\kappa(s) = \kappa_0\,\mathrm{cn}(as,p)$, where $a = \kappa_0/(2p)$ and p is the elliptic modulus. Choosing the constant of integration to be zero (one of the two possible choices that guarantee that X is well-defined) we compute $\int^s \kappa ds' = 2\arcsin(p\,\mathrm{sn}(as,p))$, and $\sin(\int^s \kappa ds') = 2p\,\mathrm{sn}(as,p)\,\mathrm{dn}(as,p) = -\kappa_s/a^2$. (In the remainder of the paper, notation for elliptic functions and integrals is taken from [1].) It follows that

$$Y(\kappa) = \kappa_s\left(\mu + \frac{\tau^2}{a^2}\right)$$

which vanishes if one selects $\mu = -4\tau^2/a^2$. This shows that $Y = -4(\tau p/\kappa_0)^2\,T + X$ is a Killing field, and the infinitesimal Bäcklund transformation is a slide-Killing vector field.

4. Double Bäcklund transformations and elastic rods

The *double Bäcklund transformation* is obtained by first performing a gauge transformation (4), but calculated using a solution of (3) for $\lambda = \nu$ a complex number instead, and then following this by a gauge transformation using a related solution of (3) for $\lambda = \bar\nu$. Specifically,

$$\Psi^{(2)} = \widetilde{G}G\Psi, \tag{10}$$

where G is as in (5), and

$$\widetilde{G} = \frac{1}{\sqrt{\lambda^2 - \bar{\nu}^2}} \left(\lambda I + \bar{\nu} \begin{pmatrix} \cos\tilde{\beta} & \sin\tilde{\beta} \\ \sin\tilde{\beta} & -\cos\tilde{\beta} \end{pmatrix} \right),$$

with

$$\tilde{\beta} = i \log\left(\zeta \frac{\nu|\zeta|^2 - \bar{\nu}}{\bar{\nu}|\zeta|^2 - \nu} \right), \qquad \zeta = -e^{-i\beta}.$$

The new curve is then obtained from $\Psi^{(2)}$ as in (2).

The double Bäcklund transformation of a closed constant torsion curve of length L can be made to be closed of length kL (k a positive integer) if the solution ψ of (3) is chosen to be kL-periodic, up to scalar multiple. For generic values of ν, there exist only two such solutions, up to multiple, and for these the new curve is congruent to the old one [3, 6]. However, any ψ may be used when ν is chosen to be a double root of the discriminant

$$\Delta_k(\lambda) = \left(\operatorname{tr} \Psi(kL; \lambda)\right)^2 - 4. \tag{11}$$

In this section, we give an explicit count of these double points when γ is a closed elastic rod centerline, and curvature is a multiple of an elliptic cosine function of modulus p. We exclude those ν-values which are pure imaginary, for which (10) is trivial.

Setting $\kappa = \kappa_0 \operatorname{cn}(x)$, where $x = as$ and $a = \kappa_0/(2p)$, we obtain a scale-invariant form of (3),

$$\frac{d\psi}{dx} = \frac{1}{2} \begin{pmatrix} q & 2p \operatorname{cn} x \\ -2p \operatorname{cn} x & -q \end{pmatrix} \psi, \qquad q = \lambda/a. \tag{12}$$

Suppose that $L = 2nK/a$ for some positive integer n. Then $\Delta_k(\lambda) = -4\sin^2(knK\Lambda)$, where

$$\Lambda = -iZ(\arcsin\alpha) + i\alpha pq, \qquad \alpha = \frac{\sqrt{p^{-2}(1+q^2)^2 - 4q^2}}{1 - q^2} \tag{13}$$

and Z denotes the Jacobi zeta function (cf. [3], §2.4). Since the symmetries of the linear system imply that $-q$ and $\bar{q}$ give the same curves as q, we will only count double roots in one quadrant of the complex plane.

Proposition 4.1. For a closed constant torsion elastic rod centerline, composed of n congruent segments forming an (m, n) torus knot, the discriminant (11) has $|m|k - 1$ double roots in the interior of each quadrant.

Proof. The formula for Λ is simplified when we let

$$q = \frac{\operatorname{sn} v}{1 + \operatorname{cn} v}, \tag{14}$$

where we will confine v to the domain $[-K, K) \times [-2iK', 2iK')$ in the complex plane, on which q is a one-to-one function of v. The branches of the square root in (13) meet at $v = \pm K \pm iK'$. Since the discriminant formula was obtained in [3] by analytic continuation from $i\lambda = \tau \in \mathbb{R}$, we choose the branch that is positive when q and v are pure imaginary, giving $\alpha = \operatorname{dn} v/(p \operatorname{cn} v) = \operatorname{sn}(v + K + iK')$. Furthermore, in order for $\Lambda(p,q)$ to be positive and continuous at $q = 0$, we must use $\arcsin \alpha = \pi - \operatorname{am}(v + K + iK')$, giving

$$\begin{aligned}\Lambda &= i\left[Z(v + K + iK') + \frac{\operatorname{dn} v(1 - \operatorname{cn} v)}{\operatorname{cn} v \operatorname{sn} v}\right] \\ &= iZ(w) + p \operatorname{cn} w, \qquad w = v + iK'\end{aligned}$$

This shows Λ to be a holomorphic function of v.

On the domain we have chosen for v, Λ is real along the imaginary axis and along arcs bifurcating out from that axis at points where $\operatorname{cn} v = E/(K - E)$. The only critical points of Λ, as a function of v, are at these bifurcation points, and so Λ is monotone along the arcs. At the bifurcation point along the negative imaginary axis, $i\lambda = iaq$ is the torsion of the closed elastic rod, and $2K\Lambda$ equals the change in cylindrical coordinate θ over one segment of the rod (see [3], eq. (9)), so that $knK\Lambda = km\pi$. Since Λ approaches zero at the branch points $v = \pm K - iK'$, we see that $0 < knK\Lambda < km\pi$ along these arcs. □

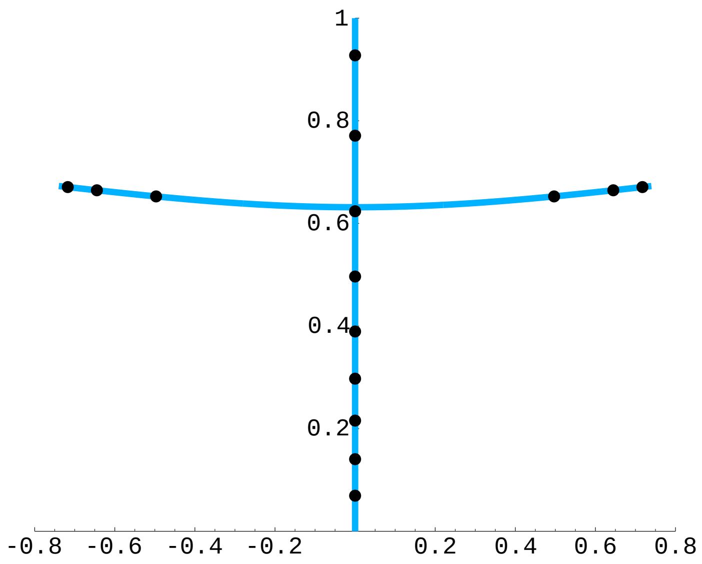

FIGURE 3. The spectrum of the linear system (12) for a (2,5)-torus knot. Only the upper half of the complex q-plane is shown, since the spectrum has symmetry $q \to \bar{q}$. The shaded curves represent the continuous spectrum, consisting of the imaginary axis and a single complex band ending at the points $p \pm ip'$. The dots locate the complex double points (non-degenerate zeros of the discriminant). In this case $p = .73983$, $k = 2$, giving 3 complex double points along the complex band of spectrum in each of the quadrants, in agreement with Proposition 4.1, together with an infinite number of imaginary double points.

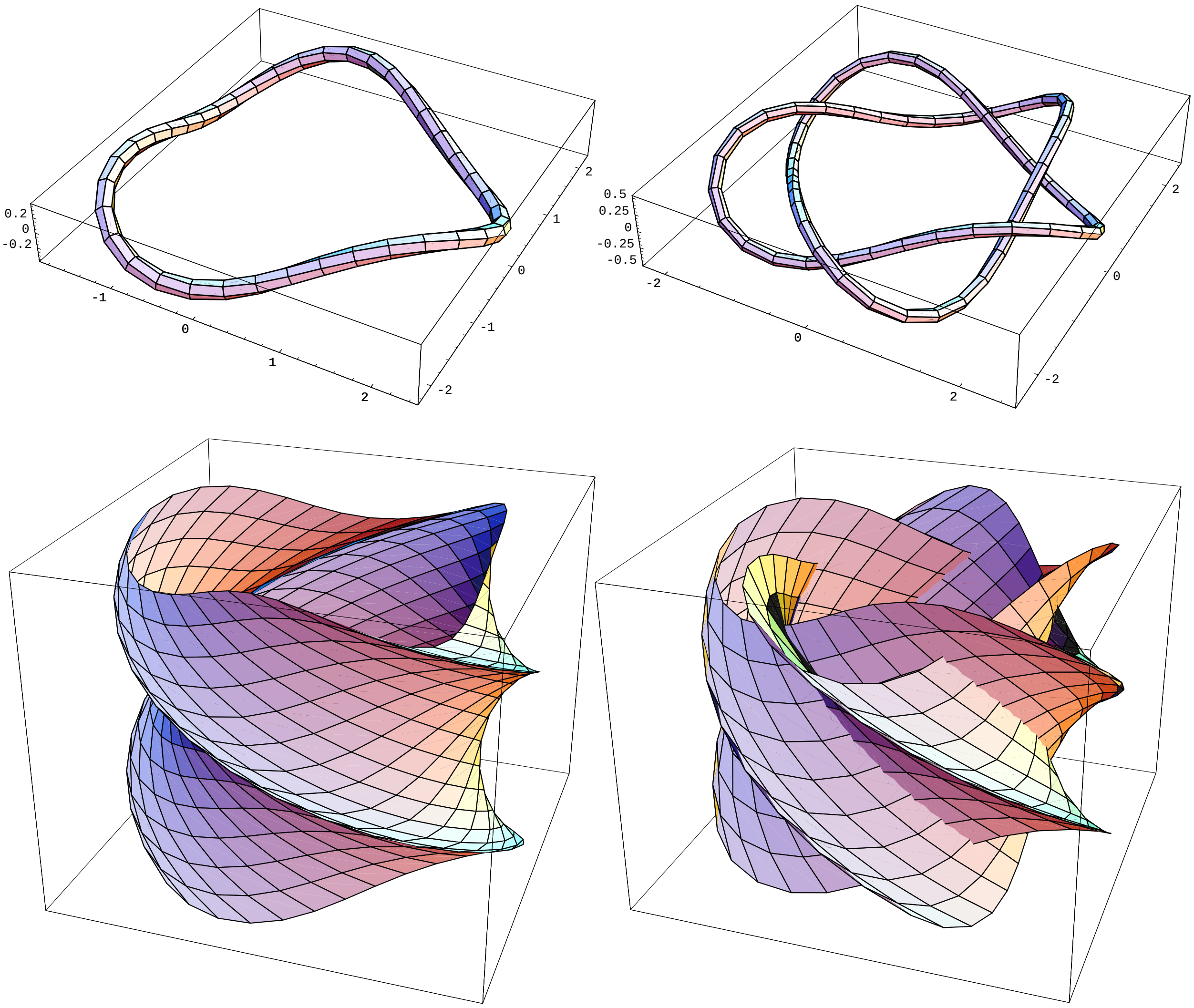

FIGURE 4. Pseudospherical surfaces generated by the elastic rod centerlines at top.

5. BÄCKLUND TRANSFORMATIONS AND SINE-GORDON EVOLUTION

The curvature $\kappa = \kappa_0 \operatorname{cn}(as, p)$, $a = k_0/(2p)$, of a constant torsion elastic rod centerline gives initial data $u(s, 0)$ for the following solution of the sine-Gordon equation:

$$u(s,t) = 2 \arcsin(p \operatorname{sn} x), \qquad x = as - t/a, \tag{15}$$

the connection being that $u_s = \kappa$ at time zero. We will explain how (15) is associated to a pseudospherical surface, using special properties of this solution, and then explain the general construction.

We can think of the surface as representing an evolution of the s-coordinate curves over time t. If T, N, B are Frenet frame vectors for one such curve, then the unit vector $T \cos u - N \sin u$ points in the direction of increasing time. For the one-phase solution represented by (15), $\partial_t + a^{-2}\partial_s$ doesn't change the curvature or the torsion of the s-curves, and so must represent the restriction to the curve of a Killing field. In fact, in terms of the vector fields defined in §3,

$$\partial_t = \frac{1}{a^2\tau^2}\left(V_2 + \tau V_1 - \tau^2 T\right).$$

Thus, up to a tangential component, these elastic rod centerlines move by a screw motion, generated by $V_2 + \tau V_1$, as we move across the pseudospherical surface. Some of these surfaces are shown in Figure 4.

One can more directly associate a surface (or, an evolving curve) to a solution of sine-Gordon using a matrix-valued solution $\Psi(s,t;\lambda)$ to (1) which depends smoothly on λ. As in (2), we let

$$\gamma(s,t) = i\Psi^{-1}\left.\frac{d\Psi}{d\lambda}\right|_{\lambda=-i\tau}.$$

Then $\gamma(s,t)$ sweeps out a pseudospherical surface, along which the s-curves have torsion τ and the t-curves have torsion $-\tau$. Of course, to carry out this construction one needs to know the matrix Ψ as a function of λ.

For (15), Ψ can be expressed in terms of theta functions (see [1]), as follows. As in (14), let $v \in \mathbb{C}$ be such that $\lambda/a = \operatorname{sn} v/(1+\operatorname{cn} v)$, and let

$$c = \frac{\pi}{2}\left(\frac{v}{2K} - 1\right).$$

Then our matrix solution is

$$\Psi(s,t;v) = \begin{pmatrix} \dfrac{\theta_0(z+c)}{\theta_1(c)\theta_0(z)} & \dfrac{\theta_2(z-c)}{\theta_3(c)\theta_0(z)} \\ \dfrac{\theta_2(z+c)}{\theta_3(c)\theta_0(z)} & \dfrac{\theta_0(z-c)}{\theta_1(c)\theta_0(z)} \end{pmatrix} \begin{pmatrix} e^{bz+\delta t} & 0 \\ 0 & e^{-bz-\delta t} \end{pmatrix} \tag{16}$$

where

$$z = \frac{\pi x}{2K} = \frac{\pi}{2K}(as - t/a), \qquad b = \frac{2K}{\pi}\frac{\lambda}{a} - \frac{\theta_3'(c)}{\theta_3(c)}, \qquad \delta = \frac{\operatorname{dn} v}{2a \operatorname{sn} v}.$$

As mentioned in the introduction, our Ψ depends on two linearly independent phases, z and $bz + \delta t$, and thus we call the evolving curves *two-phase* . Notice, however that the second phase only enters in the different scale factors of the first and second columns of Ψ.

When ν is chosen to be a double root of the discriminant, any complex (nonzero) linear combination

$$\psi = \Psi\left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right)$$

of the columns of Ψ leads to a closed curve; since scaling ψ does not affect the transformation, there is a Riemann sphere's worth of such transformations, parametrized by $\omega = c_2/c_1$. It is known (cf. [3], Figure 2.4) that the knot type of the transformed curve may change as ω varies.

The specific form of (16) leads to two conclusions about the time evolution of double Bäcklund transformations of constant torsion elastic rod centerlines.

1. The two columns of (16) are each solutions of period L (up to scalar multiple), and lead to curves congruent to the original rod. Therefore, as $t \to \pm\infty$, any other solution ψ is asymptotically a multiple of one of the columns of (16), and the curve is asymptotic to a (possibly k-times-covered) copy of the original rod.
2. Let $\psi(s,t;\omega)$ denote a linear combination of columns of Ψ using $c_2/c_1 = \omega$. Then

$$\psi(s,t;\omega) = \psi(s,0;e^{2\delta t}\omega),$$

up to scalar multiple and a translation in s. Therefore, evolution in time gives the same curves as modifying ω at time zero, which will in general change the knot type.

In Figure 5, we show the time evolution of a closed double Bäcklund transformation of the (2,5) rod. The corresponding pseudospherical surface is shown in Figure 6. Notice that the knot type changes with time.

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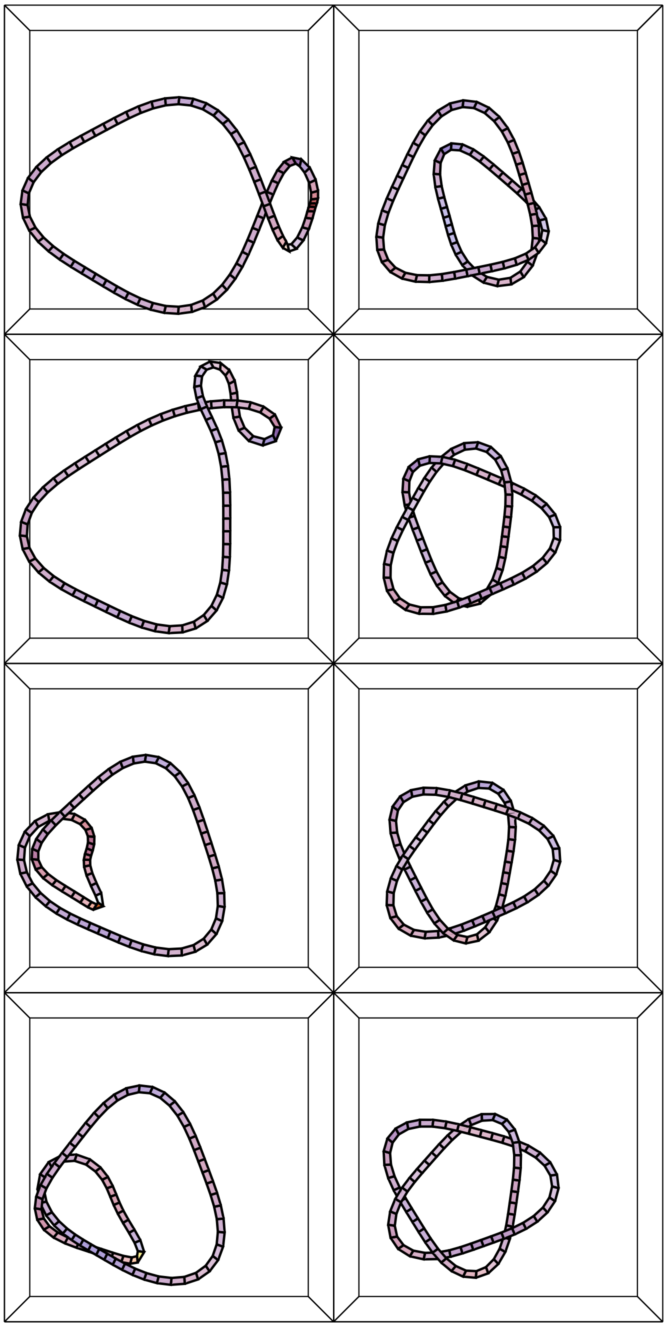

FIGURE 5. Time evolution of a double Bäcklund transformation, seen from above, displayed without the rotation of the original elastic rod (on the right in figure 4). Frames are shown for $t = 0, .6, 1.8, 2.2$ on left, $t = 3.4, 4.8, 5.6, 6.4$ on right. Knot type changes from unknotted to trefoil to (2,5) torus knot.

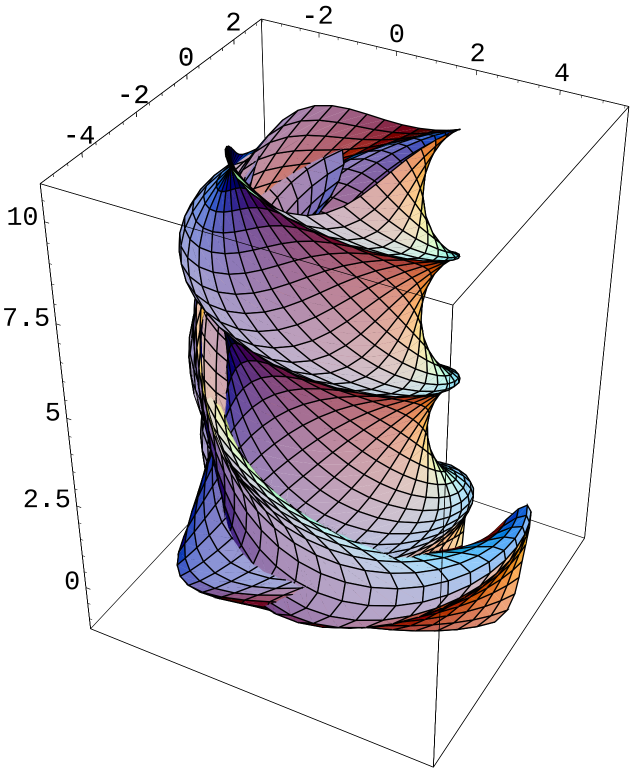

FIGURE 6. Pseudospherical surface swept out by curves in the previous figure.